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NEAR-AXIS ASYMPTOTE OF THE BATHTUB-TYPE INVISCID VORTICAL FLOWS

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1. Introduction

In the present work, we investigate the asymptotic behavior of a bathtub-type vortical flow near the axis. Special attention is paid to the most interesting case when the intensity of axial vorticity in the flow is high. The axisymmetric flow with the strong vorticity was previously investigated by Lundgren [1], who assumed that the strong vortex approximation [2] is valid everywhere in the flow. In this case, the asymptote near the axis is determined by the velocity profile in the drain, which is not known and had to be presumed in [1]. In addition, the solution obtained in [1] can not satisfy the conventional conditions in the drain requiring zero value of the radial component of the velocity. Comparisons with the experiments [3] and the numerical analysis [4] indicate that the strong vortex approximation is not likely to be valid in the immediate vicinity of the drain where the flow has to be rapidly adjusted to the conditions in the draining pipe. The present asymptotic analysis focuses on the vorticity evolution in the vicinity of the axis without attempting to stretch the strong vortex approximation to the drain. This analysis involves the higher order terms and indicates that the stream function of the strong vortex flow behaves as $\psi \sim r^{4/3}z$ near the axis. This asymptote is confirmed by the numerical results obtained for the bathtub flow [4]. As in [1], we assume that the Reynolds number is high so that the vortex flow remains inviscid except, may be, for thin boundary layers, which are not specifically considered here. The effects of viscosity and of the air dip on the strong vortex flow near the axis are standard and the reader can be referred to [1] for details.

2.Vortical flow: the governing equations

Simple observations indicate that the bathtub vortex flow is axisymmetric, laminar and, since the typical Reynolds number in the flow is very high, inviscid. The equations controlling axisymmetric unsteady flows of inviscid fluid [5] can be written in the following dimensionless form

$$\frac{1}{R}\frac{\partial^2\Psi}{\partial Z^2}+\frac{\partial}{\partial R}\frac{1}{R}\frac{\partial\Psi}{\partial R}=-\Omega_\theta\equiv\frac{\partial V_z}{\partial R}-\frac{\partial V_r}{\partial Z},\qquad V_z=\frac{1}{R}\frac{\partial\Psi}{\partial R},\qquad V_r=-\frac{1}{R}\frac{\partial\Psi}{\partial Z}\tag{1}$$

$$\frac{D\Gamma}{DT}=0, \qquad \Omega_r=-\frac{1}{\mathrm{St}}\,\frac{1}{R}\,\frac{\partial\Gamma}{\partial Z}, \qquad \Omega_z=\frac{1}{\mathrm{St}}\,\frac{1}{R}\,\frac{\partial\Gamma}{\partial R} \tag{2}$$

$$\frac{D(\Omega_\theta / R)}{DT}=\frac{1}{\mathrm{Rs}^2}\vec{\Omega}\cdot\nabla\left(\frac{\Gamma}{R^2}\right)=-\frac{2}{\mathrm{Rs}^2}\frac{\Gamma\Omega_r}{R^3} \tag{3}$$

$$\frac{D\Omega_z}{DT}=\Omega_z\frac{\partial V_z}{\partial Z}+\Omega_r\frac{\partial V_z}{\partial R}, \qquad \frac{D\Omega_r}{DT}=\Omega_z\frac{\partial V_r}{\partial Z}+\Omega_r\frac{\partial V_r}{\partial R} \tag{4}$$

where

$$\mathrm{Rs}\equiv\frac{v_0}{\sqrt{\gamma_0\omega_0}}, \qquad \mathrm{St}\equiv\frac{r_0^2\omega_0}{\gamma_0}, \qquad \frac{D}{DT}\equiv\mathrm{St}\frac{\partial}{\partial T}+V_z\frac{\partial}{\partial Z}+V_r\frac{\partial}{\partial R} \tag{5}$$

represent the Rossby number, the Strouhal number and the substantial derivative respectively; capital letters denote the normalized values: $R = r/r_0$, $Z = z/r_0$, $V_r = v_r/v_0$, $V_z = v_z/v_0$, $\Psi = \psi/(v_0 r_0^2)$, $\Omega_\theta = \omega_\theta r_0/v_0$, $\Gamma = \gamma/\gamma_0$, $\Omega_r = \omega_r/\omega_0$, $\Omega_z = \omega_z/\omega_0$, $T = t/t_0$; the circulation γ is introduced as $\gamma \equiv v_\theta r$; the choice of the cylindrical coordinates r and z is shown in Figure 1; v_0, r_0, ω_0 and γ_0 are the selected constant characteristic values of the region under consideration; z_0 is the same as r_0. The dimensional form of the governing equations is not given since one can easily obtain these equations by formally putting St=1 and Rs=1. We also note that vorticity evolution equations (4), which are used autonomously in the paper, can be obtained from (2). The choice of the characteristic time, $t_0 = \gamma_0/(v_0 r_0 \omega_0)$, needs some remarks. The characteristic time is determined from the circulation transport equations (2) which can be written in the form $\partial\gamma/\partial t = v_r\omega_z r - v_z\omega_r r$ resulting in the following estimation $\gamma_0/t_0 \sim v_0 r_0 \omega_0$. Initially, when the bathtub flow represents a solid-body rotation, the circulation and vorticity are linked by $\gamma = \omega_z r^2/2$ so that γ_0 and ω_0 are dependent. After a very short period of time, when the fluid particles from more remote regions arrive into the near-axis region the value of γ near the axis is much larger than its initial value. Hence, γ_0 represents an independent parameter and, with exception of the very short initial period, the time evolution of the flow is slow and the Strouhal number is small: St << 1 (although we should emphasize that the flow remains unsteady).

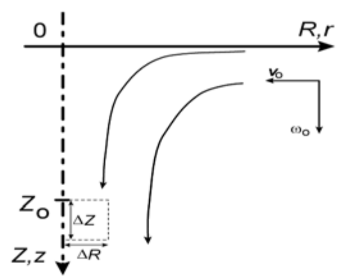

FIG.1
Schematic of the vortical flow near the axis

3. The strong vortex approximation

In this section, we investigate the asymptotic behavior of the flow near the axis assuming that the rotation in the flow is fast: Rs<<1. The strong vortex expansions

$$\Gamma = \Gamma_0 + \mathrm{St}\,\Gamma_1 + \mathrm{St}\,\mathrm{Rs}\,\Gamma_2 + ..., \qquad V_r = V_{r0} + \mathrm{St}\,V_{r1} + \dots, \qquad \Omega_r = \Omega_{r0} + \mathrm{St}\,\Omega_{r1} + \dots,$$
$$V_z = V_{z0} + \dots, \qquad \Psi_z = \Psi_{z0} + \dots, \qquad \Omega_z = \Omega_{z0} + \dots, \qquad \Omega_\theta = \Omega_{\theta 0} + \dots$$

are formulated here for small values of St and Rs. Substitution of these expansions into system (1)-(4) yields

$$\Psi_0 = F_0(R,T) + ZF_1(R,T), \qquad \Omega_{\theta 0} = -\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial \Psi_0}{\partial R}\right), \qquad V_{z0} = \frac{1}{R}\frac{\partial \Psi_0}{\partial R}, \qquad V_{r0} = -\frac{1}{R}\frac{\partial \Psi_0}{\partial Z} \tag{6}$$

$$\Gamma_0 = \Gamma_0(T), \quad \Gamma_1 = \Gamma_1(R,T), \quad \frac{\partial \Gamma_0}{\partial T} + V_{r0}\frac{\partial \Gamma_1}{\partial R} = 0, \qquad \Omega_{r0} = 0, \qquad \Omega_{z0} = \frac{1}{R}\frac{\partial \Gamma_1}{\partial R} \tag{7}$$

$$\Omega_{z0} = \Omega_{z0}(R,T), \qquad V_{r0}\frac{\partial \Omega_{z0}}{\partial R} = \Omega_{z0}\frac{\partial V_{z0}}{\partial Z} \tag{8}$$

The leading order functional representation of Ψ in (6) was originally obtained by Einstein and Li [2]. The functions F_0 and F_1 are, generally, arbitrary functions which are to be determined. The higher order equations

$$-2\frac{\Gamma_0 \Omega_{r1}}{R^3} = V_{z0}\frac{\partial \Omega_{\theta 0} R^{-1}}{\partial Z} + V_{r0}\frac{\partial \Omega_{\theta 0} R^{-1}}{\partial R} \tag{9}$$

$$-V_{r1}\frac{\partial \Gamma_1}{\partial R} = V_{z0}\frac{\partial \Gamma_2}{\partial Z} + V_{r0}\frac{\partial \Gamma_2}{\partial R}, \qquad -\frac{1}{R}\frac{\partial \Gamma_2}{\partial Z} = \Omega_{r1} \tag{10}$$

which specify the values of Ω_{r1} and V_{r1}, are used in the present analysis.

The value of F_0 can be found from boundary conditions. Since the line $Z = 0$ is, as it shown in Figure 1, a streamline Ψ = const, we can put $F_0 = 0$ in (6). The shape of F_1 near the axis of the flow is not known and was presumed by Lundgren [1] who believes that the strong vortex approximation is valid everywhere in the flow and F_1 determines the axial velocity profile in the drain (which is unknown and has to be presumed). This is not likely to be the case in a realistic flow. First, the strong vortex approximation (6) can not satisfy the conventional conditions in the drain $V_r = 0$. Second, experiments of [3] indicate that the flow experiences rapid acceleration near the drain and this does not comply with (6). In the present work, our goal is to determine the near-axis asymptote of Ψ by analyzing evolution of vorticity in the flow while considering the terms of higher order. We assume that $\Psi \sim R^{\alpha}Z$ (that is $F_1 \sim R^{\alpha}$) near the axis. The value α is not known a priory. The velocity, vorticity and circulation are determined by (6)-(10)

$$\Psi_0 = C_1 R^{\alpha}, \qquad V_{r0} = -C_1 R^{\alpha-1}, \qquad V_{z0} = C_1 \alpha R^{\alpha-2} Z, \qquad \Omega_{\theta 0} = -C_1 \alpha(\alpha-2) R^{\alpha-3} Z \tag{11}$$

$$\Omega_{z0} = -\frac{\Gamma_0'}{V_{r0}} = \frac{\Gamma_0'}{C_1 R^{\alpha}}, \qquad \Gamma_1 = -\frac{\Gamma_0'}{(\alpha-2) C_1 R^{\alpha-2}}, \qquad \Gamma_0' \equiv \frac{d\,\Gamma_0}{dT} \tag{12}$$

$$\Omega_{r1} = 2\alpha(\alpha-2)\frac{C_1^2 R^{2\alpha-3} Z}{\Gamma_0}, \qquad \Gamma_2 = -\alpha(\alpha-2)\frac{C_1^2 R^{2\alpha-2} Z^2}{\Gamma_0} \tag{13}$$

$$V_{r1} = 2\alpha(\alpha-2) C_1^4 \frac{R^{4\alpha-5} Z^2}{\Gamma_0' \Gamma_0} \qquad \eta \equiv \left|\frac{V_{r1}}{V_{r0}}\right| = 2\alpha(\alpha-2) C_1^3 \frac{R^{3\alpha-4} Z^2}{\Gamma_0' \Gamma_0} \tag{14}$$

where the functions are calculated: V_{r0}, V_{z0} and $\Omega_{\theta 0}$ – from (6); Γ_1 and Ω_{z0} – from (7); Ω_{r1} – from (9); Γ_2 and V_{r1} – from (10). The ratio η is also evaluated in (14). As it can be seen from (14), the value of $\alpha = 4/3$ corresponds to a special, limiting regime of the strong vortex. Indeed, if $\alpha < 4/3$ then $\eta \to \infty$ as $R \to 0$. Hence, the leading order structure of the flow, determined by Ψ_0 and V_{r0} in (6), can not be sustained by the flow. Physically, this means that the vorticity, which can be generated by (3), is not sufficient to keep $\alpha < 4/3$. We also note that at $\alpha < 2$, the flow has a singularity at the axis: $V_{z0} \to \infty$ as $R \to 0$. Faster downward velocities are, indeed, observed near the axis of the bathtub-type flows but, of course, this velocity can not be infinite. Physically, the influence of viscosity or presence of the air dip should be taken into account in a very thin zone near the axis. The reader can be referred to [1] where these matters are considered. It is possible to demonstrate that the singularity of the inviscid solution disappears under influence of viscosity in the viscous core. Thus, any $\alpha < 2$ can represent the near-axis asymptote only in the inertial, inviscid region.

4. Disturbing the vortical flow

In this section we analyze how the strong vortex flow can react on disturbances arriving from the peripheral regions of the flow. Batchelor [5] considered the solid body rotation in a uniform axisymmetric inviscid steady pipe flow and demonstrated that, if the Rossby number is smaller than a certain critical value, then even small variations of the pipe diameter can cause dramatic changes in the flow (this happens when the function specifying the disturbance takes zero values at the flow boundaries and becomes an eigenfunction of the problem). Physically, this means that the nature of the flow is altered and the flow is likely to loose stability and become turbulent. The results of [5] can be generalized for the flow considered here. The specific case of solid-body rotation considered in [5] results in a linear equations for the stream function. In a more general case, the flow is controlled by non-linear equations which need to be linearized. (Note that unlike the scalar transport, the vorticity transport involves nonlinear interactions with the velocity field [5].) We consider a small axial region at certain distance from the stagnation point $\Delta Z << Z_0$, $\Delta R << Z_0$ as shown in Figure 1. In this region, the flow is almost uniform and can be approximated

$$\Psi \to \Psi_0(R) = C_1 R^\alpha Z_0, \qquad V_z \to V_{z0}(R) = C_1 \alpha R^{\alpha-2} Z_0, \qquad V_r \to 0 \tag{15}$$

$$\Omega_\theta \to \Omega_{\theta 0}(R) = -C_1 \alpha(\alpha-2) R^{\alpha-3} Z_0 \tag{16}$$

$$\Omega_z \to \Omega_{z0}(R) = \frac{\Gamma_0'}{C_1 R^\alpha}, \qquad \Omega_r \to 0, \qquad \Gamma(R) \to \Gamma_0 - \mathrm{St}\frac{\Gamma_0'}{(\alpha-2) C_1 R^{\alpha-2}} \tag{17}$$

The value Z_0 is treated here as a constant. Equations (15)-(17) represent exact solution of the system (1)-(4). We consider small disturbances

$$\Psi_z = \Psi_{z0} + \varepsilon \Psi_{z1} + \ldots, \qquad V_z = V_{z0} + \ldots, \qquad V_r = \varepsilon V_{r1} + \ldots, \qquad \Omega_\theta = \Omega_{\theta 0} + \varepsilon \Omega_{\theta 1} + \ldots$$
$$\Omega_z = \Omega_{z0} + \ldots, \qquad \Omega_r = \Omega_{r0} + \varepsilon \Omega_{r1} + \ldots, \qquad \Gamma = \Gamma_0 + \mathrm{St}\, \Gamma_1(R) + \ldots,$$

whose order is assumed to be $\varepsilon << 1$. No specific assumption is made in this section with respect to the Rossby number. After linearizing system (1)-(4) by neglecting the terms of order of ε^2, St and higher, we obtain

$$\frac{\partial^2 \Psi_1}{\partial Z^2} + R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial \Psi_1}{\partial R}\right) = -R\Omega_{\theta 1}, \qquad V_{r1} = -\frac{1}{R}\frac{\partial \Psi_1}{\partial Z} \tag{18}$$

$$V_{r1}\frac{\partial \Omega_{\theta 0} R^{-1}}{\partial R} + V_{z0}\frac{\partial \Omega_{\theta 1} R^{-1}}{\partial Z} = -\frac{2}{\mathrm{Rs}^2}\frac{\Gamma_0 \Omega_{r1}}{R^3}, \qquad V_{z0}\frac{\partial \Omega_{r1}}{\partial Z} = \Omega_{z0}\frac{\partial V_{r1}}{\partial Z} \tag{19}$$

The value of $\Omega_{\theta 1}$ can be found by substituting V_{r1}, determined by (18), into (19) and integrating these equations over Z while assuming that $V_{r1} = 0$ and $\Omega_{\theta 1} = 0$ when $\Psi_{z1} = 0$. The equation for the stream function takes the form

$$\frac{\partial^2 \Psi_1}{\partial Z^2} + R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial \Psi_1}{\partial R}\right) = -\Psi_1\left(\frac{2}{R^2 \mathrm{Rs}_R^2(R)} + \Phi(R)\right) \tag{20}$$

where

$$\mathrm{Rs}_{\mathrm{R}}^2(R) \equiv \mathrm{Rs}^2 \frac{V_{z0}^2}{\Gamma_0 \Omega_{z0}} = \frac{2}{BR^{2m}}, \qquad B \equiv \frac{2}{\mathrm{Rs}^2}\frac{\Gamma_0 \Gamma_0'}{\alpha^2 C_1^3 Z_0^2}, \qquad m \equiv -\frac{3\alpha - 4}{2}$$

$$\Phi(R) \equiv \frac{R}{V_{z0}} \frac{\partial \Omega_{\theta 0} R^{-1}}{\partial R} = -\frac{\beta}{R^2}, \qquad \beta \equiv (\alpha - 2)(\alpha - 4)$$

As in [5], we assume that $\partial/\partial Z << \partial/\partial R$ and obtain equation whose solution can be expressed in terms of J_ν – the Bessel function of the first kind:

$$R^2 \frac{\partial^2 \Psi_1}{\partial R^2} - R\frac{\partial \Psi_1}{\partial R} + \left(BR^{2m} - \beta\right)\Psi_1 = 0 \tag{21}$$

$$m \neq 0: \qquad \Psi_1 \sim R\, J_\nu\left(\frac{\sqrt{B}}{|m|} R^m\right), \qquad \nu = \frac{1}{|m|}\sqrt{1+\beta} \tag{22}$$

$$m = 0: \qquad \Psi_1 \sim R^\mu, \qquad \mu \equiv 1 \pm \mu_0, \quad \mu_0 \equiv \sqrt{1 + \beta - B} \tag{23}$$

Only the functions that are of smaller order at the limit $R \to \infty$ and satisfy $\Psi_1 \to 0$ as $R \to 0$ are selected. The sign of μ_0 should be selected to ensure that Real(μ) takes its minimal positive value. (If μ_0=0, then (23) can involve the logarithmic term: $\Psi_1 \sim R \ln(R)$ whose presence does not affect our analysis.)

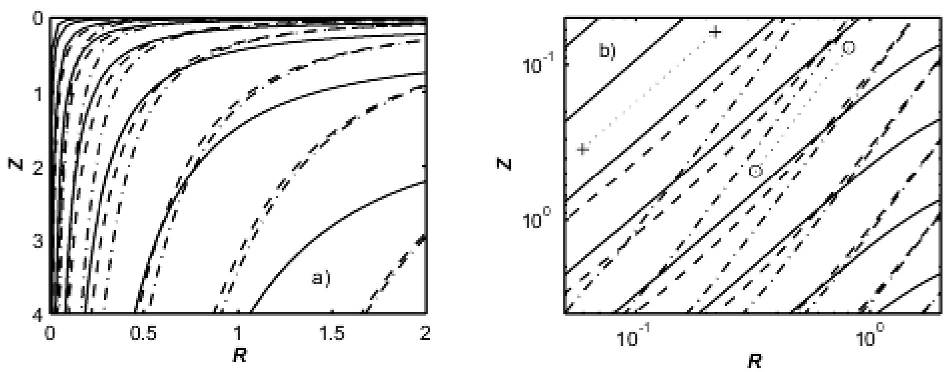

FIG. 2
Numerical analysis: streamlines in the vicinity of the axis in the bathtub flow plotted using a) normal and b) logarithmic coordinates. The Rossby number calculated using the characteristic radial velocity is: —— Rs^2=0.1; — — Rs^2=500; — - — - $\mathrm{Rs}^2=\infty$.
The lines o----o $Z \sim R^2$ and +----+ $Z \sim R^{4/3}$ are shown.

As in the previous section, the value of $\alpha = 4/3$, which corresponds to $m = 0$, represents a special, limiting regime for equation (21). Indeed, if $\alpha < 4/3$ and $m > 0$, then the far ($R \to \infty$) asymptote of (22) is given by $\Psi_1 \sim R^{1-m/2}\cos(b_1 R^m + b_2)$ where b_1 and b_2 are constants. If $\alpha > 4/3$ and $m < 0$, then $\Psi_1 \sim 1/R^b$ as $R \to \infty$ where $b = (1+\beta)^{1/2}-1$ (the oscillations occur only at smaller radii). If $\alpha = 4/3$ and $m = 0$, the exact expression for the disturbance depends on the value of B but the amplitude of the disturbance does not tend to zero as $R \to \infty$. Suitability of different values of α is now analyzed. The range $\alpha > 2$ is not realistic since $V_z \to 0$ as $R \to 0$ not consistent with the fast downward velocities observed near the axis of the flow. The range $4/3 < \alpha < 2$ corresponds to $\Psi_1 \to 0$ as $R \to \infty$. In this case, a small disturbance at a large distance from the axis can excite much larger changes in the flow near the axis. Following [5], we conclude that this flow is likely to loose stability. Hence, the range $4/3 < \alpha < 2$ must also be discarded. Finally, we note that the range $\alpha < 4/3$ is not realistic as it is shown in the previous section. Thus, only two values of the exponent, $\alpha = 2$ and $\alpha = 4/3$, appear to be acceptable. (Note that $\Psi_1 \rightarrow$const as $R\rightarrow\infty$ for $\alpha=2$ and Ψ_1 does not tend to zero as $R\rightarrow\infty$ for $\alpha=4/3$.) The value $\alpha = 2$ corresponds to $\Omega_\theta = 0$ and can be expected in the flows with weak vorticity and Rs >> 1. (Of course, α must be 2 in a potential flow when Rs=∞.) However, for the bathtub-type flows with stronger vorticity are likely to have a significant value of Ω_θ and this corresponds to $\alpha = 4/3$. The system of equations (1)-(4) has been analyzed numerically [4,6] for the bathtub flow. The enlarged near-axis region is shown in Figure 2. The value $\alpha = 2$ is observed when the vorticity level is either zero or very small. If vorticity is more significant, the region of $\alpha = 4/3$ appears near the axis. As vorticity level increases further, the region of $\alpha = 4/3$ expands in radial direction and becomes clearly distinguishable.

5. Conclusions

The near-axis asymptotic behavior of a vortical, laminar, bathtub-type flow is analyzed for large values of the Reynolds number. It is shown that only two exponents $\alpha = 2$ (the same as for the Burgers-Rott vortex) and $\alpha = 4/3$ can be expected for the near-axis inviscid asymptote of the stream function $\Psi \sim R^{\alpha}Z$ since:

- $\alpha< 4/3$, as it demonstrated in Section 3, can not be sustained by the flow;
- $4/3 < \alpha < 2$, as it is shown in Section 4, is likely to cause amplification of the disturbances near the axis and loss of stability;
- $\alpha > 2$ is not consistent with the observed non-zero downward velocities near the axis.

The value $\alpha = 2$ corresponds to weak vorticity and the value $\alpha = 4/3$ corresponds to strong vorticity in the flow. The presented analysis is supported by the numerical calculations performed for a bathtub vortical flow with different levels of vorticity as shown in Figure 2. When vorticity is strong and $\alpha = 4/3$, the local Rossby number Rs_R, introduced in (20), does not depend on R and preserves its order (it appears that Rs_R~1 while global Rs changes several orders of magnitude [4,6]). Although we note the existence of a certain similarity between the bathtub vortex flow and phenomena of much larger scales – tornadoes and cyclones –, the degree of universality of the strong vorticity laws (Rs_R~1, $\alpha = 4/3$) is yet to be examined.

6. Acknowledgment

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7. References

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