

Complex competitive systems and competitive thermodynamics.

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This publication reviews the framework of abstract competition, which is aimed at studying complex systems with competition in their generic form. Although the concept of abstract competition has been derived from a specific field — modelling of mixing in turbulent reacting flows — this concept is, generally, not attached to a specific phenomenon or application. Two classes of competition rules, transitive and intransitive, need to be distinguished. Transitive competitions are shown to be consistent (at least qualitatively) with thermodynamic principles, which allows for introduction of special competitive thermodynamics. Competitive systems can thus be characterised by thermodynamic quantities (such as competitive entropy and competitive potential), which determine that the predominant direction of evolution of the system is directed towards higher competitiveness. There is, however, an important difference: while conventional thermodynamics is constrained by its zeroth law and is fundamentally transitive, the transitivity of competitive thermodynamics depends on the transitivity of the competition rules. The analogy with conventional thermodynamics weakens as competitive systems become more intransitive, while strongly intransitive competitions can display types of behaviour associated with complexity: competitive cooperation and leaping cycles. Results of simulations demonstrating complex behaviour in abstract competitions are presented in the electronic supplementary material (ESM).

Keywords: mixing, turbulence, evolving competitive systems, complexity, non-conventional and non-equilibrium thermodynamics

1. Introduction

Thermodynamics occupies a special place among other physical sciences by postulating irreversibility of the surrounding world as its principal law. It is not a surprise that thermodynamic principles are often invoked in relation to various evolutionary processes in which irreversibility plays a prominent role. While thermodynamics has proven to be successful in explaining the common trend of moving towards equilibrium states, the complexity observed in many non-equilibrium phenomena may seem to be unnecessary if viewed from a thermodynamic perspective. Turbulent fluid motions, the existence of life forms, the complexities of technological development and many other processes involving a substantial degree of coherent

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behaviour can be mentioned as phenomena that can hardly be explained by the known trend of entropy to increase in time. On one hand there are no violations of the first two laws of thermodynamics known to modern science; on the other hand it is not clear why nature appears to be more complex than it has to be in order to comply with these laws. Erwin Schrödinger in his famous essay “What is life?” [1] articulated that life forms operate in perfect agreement with the known laws of physics and must consume exergy (negative entropy) from external sources to support their existence. Schrödinger [1] has also made a prediction that new laws of nature explaining the complex working of living organisms will be discovered in the future. This prediction has not come true yet but, if it ever will, it seems most likely that the unknown laws will have something to do with studies of complexity.

Although complex systems can be expected to possess some common properties, complexity represents a notion that is easy to understand intuitively but difficult to define in rigorous terms [2]. A general philosophical discussion of complexity might be interesting and informative but a more quantitative scientific approach to this problem needs a more specific framework. Very good examples of such frameworks are given by non-equilibrium thermodynamics [3], statistical physics [4], *algorithmic complexity* (Kolmogorov-Chaitin complexity) and algorithmic entropy [5], as well as *complex adaptive systems* [6, 7]. The present work reviews another framework – the framework of *abstract competition* – which has recently been suggested to study general principles of complex competitive systems [8–10].

Considering the apparent contradiction between the second law of thermodynamics and the high degree of organisation present in complex systems, we are also interested in the possibility (or impossibility) of effective thermodynamic characterisation of competitive systems. It appears that competitive systems (at least up to a certain level of complexity) do allow for a thermodynamic description [10]. This conclusion has profound implications: evolution of competitive systems occurs in a stochastic manner but in agreement with competitive thermodynamics. It should be noted, however, that competitive thermodynamics has its limitations when competition becomes intransitive. As an intransitive system progresses towards higher complexity associated with competitive cooperation and cyclic behaviour, the thermodynamic analogy weakens.

The idea of abstract competition was derived from the long-standing tradition of modelling turbulent combustion [11–22]. In these models, it is common to use *Pope particles* (i.e. notional particles with properties and mixing) [11, 23]. If conventional mixing is replaced by *competitive mixing* [8], these notional particles may be seen and used as computational incarnations of generic elements engaged in abstract competition. Competitive mixing can be deployed to characterise various processes: turbulent combustion, invasion waves and other related phenomena [9]. Unlike conventional conservative mixing, competitive mixing can display complex behaviour with sophisticated interdependencies, which is of special interest for this review.

The work over this review has stimulated another discovery: abstract competition seems to be a convergence point for many approaches and ideas that were developed in very different fields of science and engineering, sometimes without any apparent relevance to each other. In addition to the frameworks discussed above, we should mention, *adiabatic accessibility* [24, 25] and *Gibbs measures* [26], the *fluctuation theorem* [27] and variational principles of non-equilibrium thermody-

namics [28–31], economic *utility* [32] and economic cycles [33], Eigen’s *quasispecies model* [34] and the *Condorcet paradox* [35], as well as the theory of hydrodynamic turbulence [13, 16, 36].

2. Competitive mixing and abstract competition

Abstract competition [37] is suggested to study the principles of competition in their most generic form [8–10]. Consider a complex competitive system, which has a large number of autonomous elements engaged in competition with each other. The evolution of a competitive system involves a process of determining a winner and a loser for competition between any two elements of the system. The non-conservative properties (i.e. information) of the loser are lost while the winner duplicates its information into the resource previously occupied by the loser. The duplication process may involve random changes, which are customarily called mutations irrespective of the physical nature of the process. These mutations are predominantly negative or detrimental but can occasionally deliver a positive outcome. Interaction between the winner and loser may also involve redistribution of conservative properties, which is expected to be in favour of the winner (i.e. from the loser to the winner).

It is easy to see that abstract competition can be represented by a system of Pope particles, provided conventional conservative mixing is replaced by competitive mixing. In the present work, the terms “elements” and “particles” are used synonymously with “elements” primarily referring to competing components of general nature and “particles” to their computational implementations. We mostly focus on the non-conservative properties, which are most interesting, while limiting our consideration of conservative properties to the particles themselves (i.e. the number of particles is preserved by mixing). We also restrict our analysis to competitive mixing of couples of particles although more complicated schemes may also be considered if needed.

Let $\mathbf{y}_p$ be the set of properties associated with particle p . If particle p appears to be a winner in competition with another particle q , we may write $\mathbf{y}_q \prec \mathbf{y}_p$. On some occasions, the particles may have the same strength (i.e. $\mathbf{y}_p \simeq \mathbf{y}_q$) and no winner can be determined or the winner has to be selected randomly. Competitive exchange of information can be illustrated by the following effective reaction involving the winning particle p and the losing particle q

$$\mathbf{y}_p + \mathbf{y}_q + E \rightarrow \mathbf{y}_p + \mathbf{y}'_p, \quad \mathbf{y}_p \succ \mathbf{y}_q \quad (2.1)$$

where $\mathbf{y}'_p = \mathbf{y}_p + \mathbf{m}$ represents a mutated version of $\mathbf{y}_p$, $\mathbf{y}_p$ is stronger than $\mathbf{y}_q$ and E indicates existence of an external source of exergy that may be needed by these transformations. The mutations $\mathbf{m}$ are expected to be predominantly negative, which means that $\mathbf{y}_p \succ \mathbf{y}'_p$ is much more likely than $\mathbf{y}_p \prec \mathbf{y}'_p$.

Abstract competition deals with complex competitive systems (CCS), which share with complex adaptive systems (CAS, [6, 7]) their major premises: 1) working of a complex system is not trivially reducible to working of its elements and 2) complex systems of different physical origins should possess some in-depth similarity. CAS and CCS, however, tend to differ in the other respects. The elements of competitive systems compete rather than adapt and tend to move and mix instead of having fixed communication links typical for CAS. The long-standing Darwinian tradition of studying evolutionary systems of high complexity tends to give a higher

priority to adaptation of the system elements to their environments than to competition between the elements. CAS tend to follow this tradition, while CCS is focused on competition more than adaptation.

Competitive systems can evolve in a complex manner due to internal interactions, without any change of external conditions. In these systems, every competing element is placed in the environment of its competitors. These interactions may affect every element and at the same time may be affected by every element participating in competition. Abstract competition focuses on a joint evolution of a large number of competing elements irrespective of the physical nature of these elements. If, however, the external conditions change, a complex competitive system may respond by adapting to new conditions like any CAS is expected to behave or, if the change is large or the system is close to its stability limits, by collapsing.

Competitive mixing and abstract competition follow the idea that common properties of complex systems can be emulated in computational environments. This revolutionary idea led Eigen to formulate his quasispecies model [34]. Quasispecies evolve in time as determined by their fitness, i.e. by their specified ability to reproduce themselves. Quasispecies can compete against each other only indirectly when limitations are imposed on a common resource.

In abstract competition, the effective fitness of every element is determined by its competitors and there is no fitness specified as a pre-set parameter. Performance of an element $\mathbf{y}^\circ$ given a distribution of its competitors $f(\mathbf{y})$ can be quantified by relative ranking

$$r(\mathbf{y}^\circ, [f]) = \int_{-\infty}^{\infty} R(\mathbf{y}^\circ, \mathbf{y}) f(\mathbf{y}) d\mathbf{y}, \quad (2.2)$$

where $R(\mathbf{y}^\circ, \mathbf{y})$ is the antisymmetric index function taking values -1 , 1 or 0 when the first argument is the loser, the winner or there is a draw. The relative ranking may range $-1 \leq r \leq 1$ so that $(1 + r)/2$ specifies the probability of a win for $\mathbf{y}^\circ$ while competing with the other elements from the distribution $f(\mathbf{y})$. For example, relative ranking becomes 1 if the element $\mathbf{y}^\circ$ is stronger than all elements present in the distribution $f(\mathbf{y})$ or -1 if the element $\mathbf{y}^\circ$ is weaker than all elements present in the distribution $f(\mathbf{y})$. As the competitive system undergoes evolution, the relative ranking of every element changes since the distribution $f(\mathbf{y})$ changes as well. Under certain conditions specified by the *Debreu theorem* [38], which was introduced in economic studies more than half a century ago, competition can be equivalently characterised by absolute ranking $r_\#(\mathbf{y}^\circ) = r(\mathbf{y}^\circ, \#)$ so that

$$r_\#(\mathbf{y}_p) > r_\#(\mathbf{y}_q) \text{ if and only if } \mathbf{y}_p \succ \mathbf{y}_q \quad (2.3)$$

The absolute ranking determines how a selected element p performs with respect to another selected element q and it does not depend on f and does not change when the distribution f evolves. It should be remembered, however, that the principal conditions of the Debreu theorem — transitivity and continuity — must be satisfied to allow for the introduction of an absolute ranking. The implications of these conditions are discussed in the following sections. If fitness is interpreted as a non-specific indicator of survival and proliferation, it can be reasonably identified with the absolute ranking.

3. Competitive thermodynamics

Competitive mixing can be useful in different applications including turbulent premixed combustion [9]. The application of competitive mixing to turbulent premixed combustion follows earlier ideas of Pope and Anand [39] and results in the equations named after Fisher [40, 41] and Kolmogorov et al [40, 41] and similar to the model suggested by Bray et al. [42]. Premixed combustion is characterised by two major states of the system — the reactants and the products. If this process is interpreted as competition, the products are the winner and the reactants are the loser. The same model based on competitive mixing can be applied not only to turbulent combustion but also to a range of other processes such as invasions and simple epidemics [9, 43]. It is obvious that transition from reaction to products in combustion is driven by chemical thermodynamics. Could there be another kind of thermodynamics that generically favours winners over losers in the same way as chemical thermodynamics favours products over reactants? A positive answer to this question leads to the concept of competitive thermodynamics.

The methodology of thermodynamics requires introduction of entropy. For a system of Pope particles, the entropy can be introduced in the form [10]

$$\begin{aligned}
 S &= \underbrace{-n \int_{\infty} f(\mathbf{y}) \ln \left(\frac{f(\mathbf{y})}{A(\mathbf{y})} \right) d\mathbf{y}}_{S_c} + \underbrace{n \int_{\infty} f(\mathbf{y}) s(\mathbf{y}) d\mathbf{y}}_{S_f} - \underbrace{n \ln \left(\frac{n}{e} \right)}_{S_e} \\
 &= -n \int_{\infty} f(\mathbf{y}) \ln \left(\frac{f(\mathbf{y})}{f_0(\mathbf{y})} \right) d\mathbf{y} + n \ln \left(\frac{Z}{n} e \right), \tag{3.1}
 \end{aligned}$$

where

$$f_0(\mathbf{y}) = \frac{A(\mathbf{y})}{Z} \exp(s(\mathbf{y})) \tag{3.2}$$

is the equilibrium distribution[†] and

$$Z = \int_{\infty} A(\mathbf{y}) \exp(s(\mathbf{y})) d\mathbf{y} \tag{3.3}$$

is the partition function, while n is the number of particles, which is presumed constant. Equation (3.1) is similar to the other definitions of entropy for particle systems [44, 45]. There is a certain freedom in selecting $A(\mathbf{y})$, which makes the entropy definition invariant with respect to replacements of the variables $\mathbf{y}$. It might be convenient to simply set $A(\mathbf{y})$ to unity, but linking $A(\mathbf{y})$ to *a priori probability*, which is related to steady distributions in absence of the competition and discussed further in the manuscript, is more justified from the physical perspective. The entropy $S = S([f])$ is a functional of the distribution f which is presumed to be normalised. Entropy involves two major components: configuration entropy S_c , which is associated with chaos, and the potential entropy S_f , which is associated with the entropy potential $s(\mathbf{y})$ connected to particle properties $\mathbf{y}$. Since the particles are treated as indistinguishable (i.e. particle are the same and can be

[†] Competitive equilibriums do not generally imply achieving the equilibrium states of conventional thermodynamics. An equilibrated competitive system is in its stable steady state that is likely consume thermodynamic exergy from external sources as indicated by E in equation (2.1).

distinguished only by their properties $\mathbf{y}$), the configurational entropy is reduced by the particle exchange term $S_e = \ln(n!) \approx n \ln(n/e)$, where $e = \exp(1)$. The introduced entropy can be interpreted as being similar to the free entropy of conventional thermodynamics ($S_G = -G/T$ where G is the free energy, due to Gibbs or Helmholtz).

We may or may not know the exact mechanism behind higher competitiveness of some states as compared to the other states but, in any case, the inequality $r_{\#}(\mathbf{y}_1) > r_{\#}(\mathbf{y}_2)$ indicates that nature prefers state $\mathbf{y}_1$ to state $\mathbf{y}_2$. A thermodynamic expression for the same property is $s(\mathbf{y}_1) > s(\mathbf{y}_2)$. Hence, entropy potential s and absolute ranking $r_{\#}$ are linked to each other $s = s(r_{\#})$. Higher ranking $r_{\#}$ and higher entropy potential s recognise a greater affinity of nature towards these states, while the physical reasons responsible for this affinity may differ.

The connection between transitive continuous ordering of states by a ranking function and thermodynamic entropy is known in conventional thermodynamics under the name of adiabatic accessibility. This principle was introduced by Caratheodory [24] and recently used by Lieb and Yngvason [25] to successfully deduce the whole structure of conventional thermodynamics from this principle[†]. It is useful to note that similar methods have been under development in theoretical physics and mathematical economics for more than half a century without any knowledge or interaction between these fields. The similarity between introducing economic utility and defining entropy on the basis of adiabatic accessibility was noticed first by Candeal et. al. [46], who called it “astonishing”. This similarity seems to indicate that the concept of entropy should be also relevant to economic studies [10].

While the physical arguments in favour of connecting the entropy potential to the ranking seem convincing, constructing competitive thermodynamics requires the competitive analogue of the second law to be established. Indeed, if a competitive system evolves in isolation, its entropy S should be defined so that it always remains a non-decreasing function of time. This appears to be correct for simpler cases but as simulations progress towards cases with greater complexity the thermodynamic analogy weakens.

4. Gibbs mutations and competitive H-theorem

In this section we consider a special type of mutations — the *Gibbs mutations* — which ensure maximal consistency with thermodynamic principles. The results are applicable to both transitive and intransitive competitions. A general definition of Gibbs mutations is given Ref. [10]. Here we outline major properties of Gibbs mutations and discuss their relevance to Markov properties and Gibbs measures [26].

The Gibbs mutations are defined as strictly non-positive

$$\mathbf{y}'_p = \mathbf{y}_p + \mathbf{m} \preceq \mathbf{y}_p \quad (4.1)$$

We can consider Gibbs mutations as being represented by a sequence of a large number of small steps directed towards states with reduced competitiveness. The

[†] A similar approach, albeit based on a different quantity — *adiabatic availability*, has been previously developed by Gyftopoulos and Beretta [60]

probability of each subsequent step is independent of the proceeding steps, which essentially represents a Markov property. If a step does not occur, the whole process is terminated and the final point of the mutation is reached. The probability distribution of these mutations can be interpreted as a Gibbs measure [10].

Equation (3.1) indicates that entropy achieves its maximum $S \rightarrow n \ln(eZ/n)$ when the distribution f approaches its equilibrium f_0 . Evolution of competitive systems appears to be consistent with the following theorem:

Competitive H-theorem. *An isolated competitive system with Gibbs mutations evolves in a way that its entropy S increases in time until S reaches its maximal value at the equilibrium.*

The equations governing the evolution of competitive systems and proof of the competitive H-theorem can be found in Ref. [10]. This theorem indicates consistency of the introduced entropy with the principles of thermodynamics and is valid for both transitive and intransitive competitions.

The usefulness of the thermodynamic description can be illustrated by the following consideration: assume that two competitive systems with a fixed total number of particles $n = n_1 + n_2 = \text{const}$ are brought in contact with each other; then the total entropy $S = S_1 + S_2$ must reach its maximum when the equilibrium between the systems is established after particle exchange. Hence, the values $\chi_1 = \partial S_1 / \partial n_1 = \ln(Z_1/n_1)$ and $\chi_2 = \partial S_2 / \partial n_2 = \ln(Z_2/n_2)$ should be the same in the equilibrium. These values χ_I , $I = 1, 2$ are called *competitive potentials* and represent quantities analogous to the chemical potential of conventional thermodynamics taken with negative sign. A non-zero difference between competitive potentials $\chi_1 - \chi_2$ points to the direction of flow of the resources when two competitive systems are brought into contact.

The Great American Exchange, which took place around 3 million years ago when the Isthmus of Panama connecting two Americas was formed, may serve as an example [47]. Before the Isthmus was formed the biosystem of each continent was, presumably, close to quasi-equilibrium (see the next section). The exchange induced significant and complex changes in the biosystems. While each of these changes may have its specific cause or explanation, the methodology of thermodynamics neglects the details and recognises the overall trend behind a large number of seemingly unrelated events. On average, the North American animals performed better than their South American counterparts consistently in both native and immigrant conditions. We may infer that, at the time of the connection, the fauna of North America had somewhat higher competitive potential than the fauna of South America (which is difficult to explain by adaptation).

5. Transitive competition

When mutations deviate from Gibbs mutations, the behavior of the competitive system becomes quite different for transitive and intransitive competitions. Competition is deemed transitive when the following statement

$$\text{if } \mathbf{y}_1 \preccurlyeq \mathbf{y}_2 \text{ and } \mathbf{y}_2 \preccurlyeq \mathbf{y}_3 \text{ then } \mathbf{y}_1 \preccurlyeq \mathbf{y}_3 \quad (5.1)$$

is valid for any three states indexed here by 1, 2 and 3. Presuming continuity of the competition rules, transitive competition can be characterised by absolute

ranking (2.3). The particle that has the highest absolute rank in the distribution $r_* = r_{\#}(\mathbf{y}_*)$ is called the leading particle and its location is denoted by $\mathbf{y}_*$.

In transitive competition with negative mutations the leading particle can not lose to any other particle and can not be overtaken by any other particle – hence $\mathbf{y}_*$ remains constant and, according to the H-theorem, the distribution f converges to its equilibrium state $f_0(\mathbf{y}, \mathbf{y}_*)$. However, existence of positive mutations results in the occasional particle jumping in front of the leader. The overtaking particle then becomes a new leader and $\mathbf{y}_*(t)$ is an increasing function of time. If positive mutations are small and infrequent, the distribution f remains close to quasi-equilibrium $f_0(\mathbf{y}, \mathbf{y}_*(t))$ but progresses towards higher ranks as illustrated in Figure 1a. This process, which is named *competitive escalation*, results in increase of S in time and is perfectly consistent with competitive thermodynamics.

The behavior of competitive systems is similar to that of conventional thermodynamic systems and generally consistent with Prigogine's minimal entropy production principle [28]. The generation of entropy S decreases as the distribution $f(\mathbf{y}, \mathbf{y}_*)$ rapidly approaches its quasi-equilibrium state $f_0(\mathbf{y}, \mathbf{y}_*)$ and then takes a relatively small but positive value as $r_{\#}(\mathbf{y}_*)$ increases due to competitive escalation. The rate of competitive escalation is evaluated in and is determined by the average magnitude of positive mutations, which is linked by the *fluctuation theorem* [27] to the gradients of *a priori entropy* $\hat{s}(\mathbf{y})$ [10]. The *a priori entropy* $\hat{s}$ is related to the steady distribution of particles in absence of competition $\hat{f}_0 \sim \exp(\hat{s})$, which is named *a priori probability*, and should be distinguished from the entropy potential of the competition $s(\mathbf{y})$. The entropies $\hat{s}(\mathbf{y})$ and $s(\mathbf{y})$ increase in opposite directions since more competitive states with higher $s(\mathbf{y})$ are relatively rare and have smaller *a priori probability* $\hat{f}_0$. The applicability of the maximal entropy production principle (MEP) [31, 48] (representing a generalisation of the principle introduced by Ziegler [30]) to competitive systems remains an open question. Note that there is no contradiction between the principles of Prigogine and Ziegler as they are formulated for different constraints [31].

If mutations significantly deviate from Gibbs mutations, the thermodynamic analogy weakens but, as long as competition remains transitive, the behaviour of competitive systems is qualitatively similar to that with Gibbs mutations:

Transitive convergence theorem. *An isolated system with transitive competition and non-positive mutations converges to a unique equilibrium state in which the entropy S reaches its maximal value (for a fixed location $\mathbf{y}_*$ of the leading element).*

This theorem, which is proven in Ref. [10] under certain mathematical constraints, does not guarantee monotonic increase of entropy and existence of the detailed balance at equilibrium (these are guaranteed by the competitive H-theorem). Deviations from Gibbs mutations, however, do not affect the qualitative behaviour of a system with transitive competition: the system first rapidly approaches its quasi-equilibrium state and, provided infrequent positive mutations are present, then escalates slowly towards more competitive states and larger S . Competitive escalation, which involves gradual increase of competitiveness in competitive systems, is generally quite realistic but can not explain more sophisticated patterns of behaviour observed in the real world. An evolving transitive system may diverge as shown in Figure 1b but divergence by itself does not remove the restrictions of transitivity imposed on each of the separated subsystems. Competitive esca-

lation cannot continue indefinitely: sooner or later the competitive system should encounter restrictions associated with resource limitations, laws of physics, etc. and at this point any further development would be terminated. Evolutionary processes may still act to reduce the magnitude of mutations resulting in a slight additional increase in competitiveness (assuming that the magnitude of mutations is allowed to mutate — note that reduction of the magnitude of mutations is the only possible way for further increase in particle competitiveness in conditions when the leading particle has the maximal ranking permitted in the system). At the end, the system enters into the state of global equilibrium and cannot change, except if the external conditions are altered. Unless these external conditions become increasingly complex, the transitive system is likely to respond to this alteration by a relatively small adjustment and any significant increase in complexity remains unlikely in these conditions. A very large alteration of the environment may, of course, destroy the system. Consequently, complexity emerging within competitive systems should be associated with overcoming the major constraint imposed by transitive competition rules, i.e. with intransitivity.

6. Intransitive competition

Competition is deemed intransitive when there exist at least three states $\mathbf{y}_1$, $\mathbf{y}_2$ and $\mathbf{y}_3$ such that

$$\mathbf{y}_1 \preceq \mathbf{y}_2 \preceq \mathbf{y}_3 \prec \mathbf{y}_1 \quad (6.1)$$

For a long time, since the days of the French revolution when intransitivity was first studied by outstanding mathematician, philosopher and humanist marquis de Condorcet [35], intransitivity was mostly viewed as something unwanted or illogical [49]. Arrow's theorem [50], which is well-known in social sciences, indicates that intransitivity may even pose a problem to choice in democratic elections. Intransitivities have nevertheless become more philosophically accepted in recent times [51] and are now commonly used in physics [52], biology [53] as well as in social studies [54]. Recently published works [8, 10] indicate the importance of intransitivity for evolution of complex systems, although intransitivity may indeed have some “unwanted side effects”.

Although in the case of Gibbs mutations intransitivity does not violate the thermodynamic analogy, the effects of any deviation from Gibbs mutations combined with intransitivity can produce, depending on the conditions, very diverse patterns of behaviour. Intransitivity conventionally refers to any violation of transitivity, irrespective of intensity of these violations. Intransitive competition rules may effectively combine some transitive and intransitive properties. An example of a globally intransitive system that remains locally transitive is shown in Figure 1c. The arrow indicates the direction of increasing strength; this direction forms a circle so that k locations forming an intransitive loop $\mathbf{y}_1 \prec \mathbf{y}_2 \prec \dots \prec \mathbf{y}_k \prec \mathbf{y}_1$ can be easily nominated (the strengths of any two particles are compared in the direction of the shortest arc connecting these particles). We note that the absolute ranking $r_{\#}(\mathbf{y})$ does not exist as a conventional function and $r_{\#}(\mathbf{y})$ becomes multivalued $r_{\#}(\mathbf{y}_1) < \dots < r_{\#}(\mathbf{y}_k) < r_{\#}(\mathbf{y}_1)$. Clearly, evolution of this system would be very similar to transitive competitive escalation although the whole process becomes cyclic and the same distribution of particles (such as shown in Figure 1c) would be repeated

periodically. Competition entropy $s(\mathbf{y})$ can be introduced locally in the same way as $s(\mathbf{y})$ is introduced for transitive competition but, on a large scale, the thermodynamic quantities become multivalued $s = s(r_{\#}(\mathbf{y}))$. We may assume for simplicity that mutations are close to Gibbs mutations at every location and nevertheless find three different systems $I = 1, 2, 3$ with competition potentials χ_1 , χ_2 and χ_3 that are not transitive and symbolically satisfy the inequalities $\chi_1 < \chi_2 < \chi_3 < \chi_1$. This, indeed, must be a very unusual thermodynamics. In conventional thermodynamics, temperatures and chemical potentials are restricted by the zeroth law of thermodynamics and are fundamentally transitive.

Another example, presented in Figure 1d, shows a competition that has a transitive direction of increasing strength along the y_1 axis combined with cyclically intransitive rules (the same as in the previous example) on every plane $y_1 = \text{const}$. This system would undergo a spiral evolution including translational competitive escalation in the direction of transitive increase of the competitive strength and cyclic motions along the plane of constant y_1 . The system escalates in the transitive direction until the maximal possible value of y_1 is reached. At this moment transitive evolution in the direction of y_1 is terminated while intransitive evolution may continue its cycles indefinitely.

The main feature of intransitive competitions is the absence of clear signposts for being stronger and being weaker. The weakest particle, if left in isolation behind the distribution circulating in Figure 1c, would eventually become the strongest particle in the set. An improvement in a transitive system is always an improvement while a direction that seems to improve competitiveness of an intransitive system in the present conditions may in fact reduce its competitiveness when the overall distribution changes over time. In intransitive conditions, decisions that seem very reasonable in the current situation may appear to be detrimental in the long run (and vice versa). For example, investments in “dot-com” companies were seen as quite profitable in the 1990s but the same investments would be viewed very differently in the 2000s. A strategy in an intransitive system needs to not only be evaluated with respect to the current criteria of competitiveness, but also examined against the changes that are likely to be introduced as the system evolves. Intransitivity brings more diverse cyclic patterns of behaviour into competitive systems, while real-world competitions can be controlled by very complicated combinations of transitive and intransitive rules. The overall trend in evolution of competitive systems is the elimination of transitively weak elements, resulting in predominance of the intransitive rules in the competition (since intransitive weaknesses give better chances of survival as compared to transitive weaknesses).

7. Intransitivity and complexity

The systems considered in the previous section are intransitive but display many features of transitive competitions. This allows for application of competitive thermodynamics, which, at this point, diverges from conventional thermodynamics and involves intransitivities. We now move to consider competitions with strongly intransitive rules, where intransitive triplets (6.1) can be found in every open neighbourhood of any location $\mathbf{y}$. In these competitions, the absolute ranking cannot exist even locally. Particle strength may still be evaluated in terms of the relative ranking $r(\mathbf{y}, [f_r])$ but this ranking is highly dependent on selection of a suitable

reference distribution f_r : improvement of ranking of a particle with respect to one reference distribution may reduce the ranking of the same particle with respect to another reference distribution. In intransitive systems, assessment of progress and regress may strongly depend on the observer's perspective.

The most interesting point in examining evolution of competitive systems is the possibility or impossibility of a complex behaviour, which implies a sophisticated coordination of properties of many elements in a way that apparently involves or resembles formation of structures and a joint action of elements within each structure. The usefulness of applying the thermodynamic analogy to strongly intransitive competitions seems questionable [10]. The diversity of particle distributions associated with complexity implies localisation of these distributions and, consequently, localisation of competitive interactions in physical space. Abstract competition views intransitivity and localisation as necessarily conditions for complex behaviour, although these conditions may be insufficient to guarantee complexity (a system combining intransitivity with Gibbs mutations, which enforce relatively simple relaxation to equilibrium, may serve as a counterexample). From practical perspective, however, a sufficiently large system with strong intransitivity and localisation of competitive interactions in physical space seems to have a non-vanishing probability of developing complex behaviour. Indeed, even the most simple competitive systems that still possess the properties of strong intransitivity and localisation display clear signs of developing complexity under appropriate conditions [8].

Complex behaviour, which is associated with strong intransitivity and localisation, is accompanied by a number of indicators that have been observed in computer simulations of strongly intransitive abstract competition [8]. It should be stressed that mutations used in these simulations are purely random and do not have any purpose or coordination between particles. The results of simulations presented in the electronic supplementary material (ESM) indicate diverse patterns of behaviour, which are associated with complexity and discussed in the following subsections.

(a) *Competitive cooperation*

Competitive cooperation is the formation of structures accompanied by coordination of properties and reduction of competition intensity within the structures (as compared to intensity of competition between the structures). Survival of element properties within the structure is strongly correlated with survival of the whole structure. Complex cooperative behaviour occurs when competitive mixing is strongly intransitive and localised in physical space [8, 10]. The computer simulations of abstract competition presented in ESM involve one of the simplest possible systems that possess the properties of strong intransitivity and localisation of mixing and, indeed, these simulations display clear cooperative behaviour. The simulations indicate the following mechanisms of competitive cooperation: 1) particles have more similar properties within each structure as compared to larger differences of particle properties between the structures and 2) each structure resembles a pyramid where particles tend to compete against particles of similar relative ranks within the structure (i.e. leaders against leaders and low-rankers against low-rankers).

Coordination of properties between elements and formation of structures is the key indicator that distinguishes partially ordered from completely chaotic types

of behaviour. Cooperation thus increases complexity and, if the size of the system allows, inevitably leads to even more complex behaviours. Indeed, the formed structures may be considered in the framework of abstract competition as new elements (i.e. superelements) competing according to the integrated competition rules. These integrated competition rules are inherently intransitive and interactions between structures are localised in physical space (the *competitive Condorcet paradox* [10] indicates that the integrated competition rules are likely to be intransitive even if the underlying competition rules between primary elements are transitive). Hence the structures are likely to form groups of structures coordinating the properties of structures and the overall picture presented by abstract competition becomes more and more complicated. If the system is sufficiently large, the hierarchy of structures can grow to become very complicated. Competitive cooperation, which in most cases is seen as a very positive factor, may nevertheless have unwanted side effects: the reduction of competition associated with competitive cooperation can be accompanied by degradations.

(b) *Competitive degradation*

Competitive degradation is reduction of competitiveness that may appear in intransitive competitions under certain conditions. Competitive degradation is the process opposite to competitive escalation and, generally, is not consistent with competitive thermodynamics (see ref. [10] for further discussion).

Owing to the relativistic nature of competitiveness quantified in intransitive conditions by relative ranking $r(\mathbf{y}, [f_r])$, the presence of competitive degradations may or may not be obvious. While degradations are unambiguously present in simulations shown in the ESM, degradations can also be latent and might be superficially seen as escalations, especially when considered from a certain fixed perspective (i.e. using specific poorly selected f_r in $r(\mathbf{y}, [f_r])$). A competition process may result in increase of competitiveness with respect to one parameter, say $y^{(1)}$, and decrease of competitiveness with respect to another parameter, say $y^{(2)}$. In transitive competitions, the former always outweighs the latter and the overall absolute ranking must increase. Intransitivity makes the outcomes much less certain. Increase in $y^{(1)}$ may seem to be a competitive escalation while a minor decrease in $y^{(2)}$ may not even be noticed. However, if and when $y^{(2)}$ falls below a certain critical level, the dominant structure may become unstable with respect to large disturbances, which can be created by rare mutations, and ultimately collapse.

Competitive degradations should be clearly distinguished from *erosive degradations*: the former are the results of strict compliance with the competition rules while the latter appear due to various imperfections. Erosive degradations may exist in any competition while competitive degradations require intransitivity. In our analysis of competitions, we have assumed that information has eternal properties: that is, once obtained and not overridden in competition, information can exist forever. In the real world, however, information may be eroded and lost. Another cause of erosions may be imperfections in the competition process inducing random outcomes so that the distributions become more and more influenced by the a priori probability. Competitive degradation is different: it is a result of “perfect” competition when at every step the strongest always wins but, nevertheless, the overall competitiveness declines due to interactions of intransitivity and mutations.

This process may seem abnormal but computer simulations [8] show that it can exist under conditions of strong intransitivity and localisation.

Let us consider two examples. 1) Company managers become bureaucratic, following all formal procedures but stalling initiative and, as a result, the quality of the company's products declines. This is competitive degradation. 2) Old technical drawings lose information due to paper erosion and, because of this, the company can not sustain the quality of its products. This is erosive degradation.

It is quite obvious that erosive degradations are much easier to detect and prevent (at least theoretically) as compared to competitive degradations, whose causes are more sophisticated. It seems that competitive degradations represent a possible side effect of competitive cooperation (see ESM). Cooperations are accompanied by a reduction of competition that, in turn, may be followed by degradations.

(c) *The leaping cycle*

The *leaping cycle* is a special type of cyclic behaviour that is characterised by the emergence of a new structure, followed by its rapid growth and leapfrogging into a strong or even dominant position, stable strength or domination, decline, and a complete collapse or restricted (niche) existence. The leaping cycle differs from ordinary cycles (such as depicted in Figure 1c): leaping cycles are associated with competitive degradation and result in replacement of the old structures by new ones, while ordinary cycles are caused by local competitive escalation in conjunction with cyclic intransitivity and result in periodic repetitions of the same states.

Practically, a complicated cycle can possess features of both ordinary and leaping cycles while degradations and escalations may not be easily distinguishable. The term leaping cycle refers to a generic form of the cycle, such as observed in simulations presented in ESM. In different disciplines, cyclic behaviours similar to the leaping cycle are known under different names [37]. Technological waves with durations of around 50 years may serve as an example of cyclic behaviour. These cycles were discovered by Kondratiev [55] nearly 100 years ago as long-term oscillations in economic growth — according to this understanding the Kondratiev technological waves represent ordinary cycles. Another interpretation of the technological cycles involving irruption, rapid growth, maturity and decline is recently given by Perez [33]. This interpretation resembles the pattern of a leaping cycle much more than that of an ordinary cycle.

8. Intransitivity in turbulent flows

The presented treatment of competitive systems is generic and can be applied to systems of different physical nature. In this section, as at the beginning of this review, we consider the application of competition principles to turbulent flows. The primary goal of this section is not in suggesting a new approach for treatment of turbulent flows but in looking at existing approaches from a different perspective based on principles of abstract competition.

Eddies of different sizes form the turbulent energy cascade [13, 16, 36, 56, 57]. The development and breakdown of turbulent eddies can, in principle, be interpreted as competition between the eddies. This interpretation, however, must remain qualitative as eddies in turbulent flow tend to erode quickly and cannot be

defined as fully autonomous objects. A degree of complexity may be present in turbulent flows but it must be restricted by the high rate of eddy erosion. “Eddy competition” should be clearly distinguished from a more rigorous application of competitive mixing to turbulent reacting flows. Since the states of chemical composition are well-defined and evolve due to the well-defined combination of mixing, transport and reactions, reacting flows are much more suitable for quantitative application of competitive mixing [9].

Turbulence is a complex phenomenon which can provide examples of intransitive behaviour. Here we refer to three-dimensional turbulence since the properties of two-dimensional turbulence are quite different. Specifically, the inverse cascade of two-dimensional turbulence preserves energy and eddy vorticity, which makes this phenomenon amenable to thermodynamic description [58]. The eddies interact in three-dimensional turbulence, which is interpreted here as “competition” of eddies. It is interesting that winners and losers can be defined for interactions of eddies in turbulent flows in different ways. Energy is transferred from larger to smaller eddies, hence from the perspective of conservative properties (i.e. energy) smaller eddies are the winners. Considering non-conservative properties, it is larger eddies that statistically control the structure and intensity of the smaller eddies and the larger eddies are thus the winners. In the present consideration, we focus on information flows and treat larger eddies as winners and smaller eddies as losers.

Figure 2 shows the directions of control from the mean flow through instabilities first to large and then to smaller and smaller eddies. This figure follows Kuznetsov’s theory of energy transport from the mean shear flow into large-scale oscillations of fully developed turbulence through growth of instabilities (this theory is developed for turbulent shear flows with a mean velocity profile having at least one inflection point [59]). The scheme, which is shown in Figure 2 by solid arrows, is transitive. In transitive competitions, particles with higher absolute ranks are not directly affected by particles with lower absolute ranks, since the former are always the winners and the latter are always the losers. According to the transitive scheme, turbulence would behave in a predictable manner, because the problem can be solved at larger scales without emulating the turbulent motions at smaller scales. Many of the empirical models of turbulence are based on this assumption, although some methods (i.g. the probability density function methods) are aimed at simulating stochastic behaviour at smaller scales [16]. A number of theories dealing with the energy dissipation cascade implicitly incorporate transitivity by assuming stochastic control of turbulent motions at larger scales over turbulent motions at smaller scales [36]. While these models and theories often achieve a reasonable degree of accuracy, turbulence has proven to be a more complex phenomenon than is predicted by any existing theory or model. Turbulence is commonly believed to be the last great unsolved problem of classical physics (see the excellent collection of quotations about turbulence in Ref.[57]).

The arguments presented in this review link complexity to intransitivity. Is this link relevant to turbulence? Intransitive exchange between the Reynolds stress components in shear flows [10] presents an example of intransitivity in turbulence, but here we are interested in the turbulent eddy cascade. If the eddies are deemed to “compete” against each other for control within the cascade, intransitivity should be understood as some degree of control of smaller eddies over motions at larger scales. This reverse control is perhaps not as strong as the direct control of larger

eddies over smaller eddies but its existence would promote the smaller eddies from the status of mere followers to the status of (partially) independent players. This makes evaluation of motion at larger scales dependent on details of the stochastic events at smaller scales, which poses a problem for any models that are restricted to consideration of large scales. Figure 2 shows the direction of reverse control from smaller to larger scales by the dashed arrow. The reverse control, which may involve the influence of turbulent viscosity on the rate of growth of the instabilities in the mean flow and other mechanisms, introduces intransitivity into the turbulent cascade.

9. Conclusions

Although abstract competition was derived from ideas used in modelling of mixing in turbulent reacting flows, it pertains to study of competition principles in their most generic form. Competitive systems can be complex (CCS) and have a number of features common with complex adaptive systems (CAS). Abstract competition involves exchanging properties between competing elements and, hence, can be seen as a form of mixing (i.e. competitive mixing). This exchange distinguishes winners and losers and discriminates against the losers. The winners and losers are selected on the basis of element properties as interpreted by a set of competition rules. The degree of transitivity (or intransitivity) of the competition rules is the one of the main factors affecting the behaviours of competitive systems.

Transitive competitions result in competitive escalations, which are accompanied by gradual increase in competitiveness of the system, or in achieving a global equilibrium state that represents the final state of evolution of the system. The analogy with conventional thermodynamics is especially strong in the case of Gibbs mutations. Transitive evolutions are consistent with a thermodynamic description, at least qualitatively: the competitive entropy tends to stay the same or increase. From the perspective of competitive thermodynamics, the phenomenon of achieving higher competitiveness in transitive competitive systems is natural, in the same way that a formation of crystalline structures under appropriate conditions is natural in conventional thermodynamics. The complexity of transitive competitions, however, remains restricted and falls significantly short of explaining the complexity observed in nature.

As competition becomes more and more intransitive the analogy with conventional thermodynamics, which is fundamentally transitive, weakens. Intransitivity leads to a situation where thermodynamic functions become multivalued functions of the state of the system; for example, competitive potentials may satisfy the symbolic inequality $\chi_1 < \chi_2 < \chi_3 < \chi_1$. Needless to say, similar inequalities are impossible for temperatures or chemical potentials in conventional thermodynamics. This type of intransitivity allows for endless cyclic oscillations in competitive systems.

Complex types of behaviour are associated with stronger intransitivity, when intransitive triples can be found in the vicinity of any point, and with localisation of competitions in physical space. Although strong intransitivity and localisation may not necessarily be sufficient for development of complexity, the emergence of complexity becomes rather likely under these conditions since even the relatively simple system presented in ESM (which is essentially the simplest possible system

with strong intransitivity and localisation) displays complex behaviour. The thermodynamic analogy tends to become inapplicable as the system begins to display more complex patterns of behaviour, among which we outline competitive cooperation, competitive degradation and the leaping cycle. Competitive cooperation is formation of structures with a reduced level of competition within each structure. Competing particles or elements survive competition through a coordinated effort rather than individually. Competitive degradation, which seems to be a side effect of competitive cooperation, is an “abnormal” outcome of competition resulting in reduced competitiveness. Evolution in intransitive systems can be highly non-linear involving emergence, a leap forward to a dominant position, stagnation, degradation and, possibly, collapse. In abstract competition this type of behaviour is called a leaping cycle, although it may be related to cycles known under different names in different disciplines.

We conclude by noting the natural trend of progressing towards complexity in competitive systems. Realistic competitions are likely to include a mixture of transitive and intransitive rules. While transitive features tend to dominate at the initial stages, elimination of transitively weak elements results in competitions being restricted to regions where intransitivity prevails. Since localisation in physical space is dictated by the laws of physics, the combination of intransitivity and localisation creates the conditions needed for competitive cooperation and formation of structures. Assuming that the overall size of the system is extremely large, the principles of abstract competition can now be applied to these structures rather than to the elements. The rules for competition between the structures should also be intransitive, while their interactions remain localised in physical space. This leads to cooperative behaviour between the structures and formation of superstructures. Recursive application of the principles of abstract competition to superstructures at different levels opens the possibility of building competitive hierarchies of unlimited complexity.

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Figure 1. Evolutions in competitive systems: a) competitive escalation in a non-preferential transitive competition; b) competitive escalation in transitive competition with particle isolations causing divergence of distributions; c) evolution in a system with cyclic intransitivity; d) spiral evolution in a more complex system combining transitive and intransitive properties. The arrows with the oval-shaped starting points show the direction of competitive escalation and increasing ranking.

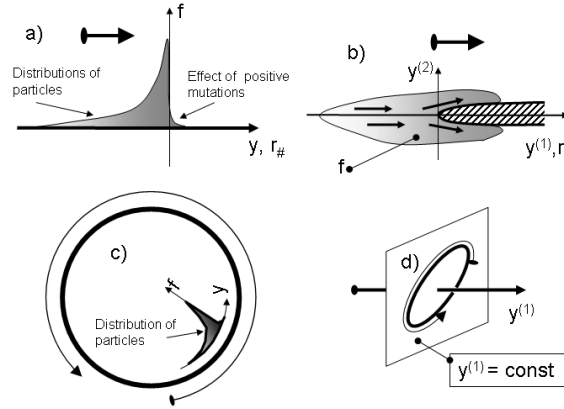


Figure 2. Presence of intransitivity in the turbulent cascade.

