A small disturbance in the strong vortex flow

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A small disturbance in the axisymmetric, bathtub-like flow with strong vorticity is considered and the asymptotic representation of the solution is found. It is shown that if the disturbance is smaller than a certain critical scale, the conventional strong vortex approximation cannot describe the field generated by the disturbance not only in the vicinity of the disturbance but also at the distances much larger than the critical scale. © *2001 American Institute of Physics.*
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An axisymmetric flow with strong axial vorticity represents a reasonable model for various phenomena of very different scales ranging from the size of a bathtub vortex to the scale of atmospheric cyclones. The strong vortex approximation, which corresponds to small values of the Rossby number, was originated by Einstein and Li[1] and confirmed by the asymptotic analysis of the Navier–Stokes and Euler equations introduced by Lewellen[2] and Lundgren.[3] Their major result is represented by the following leading order equation for the stream function ψ:

$$\psi = f_0(r,t) + z f_1(r,t), \tag{1}$$

where f_0 and f_1 are functions which are restricted only by the boundary conditions imposed on the flow, and r and z are the radial and axial coordinates. In this flow, the radial velocity $v_r = -r^{-1}\partial\psi/\partial z$ does not depend on z and the axial velocity $v_z = r^{-1}\partial\psi/\partial r$ depends on z linearly.

The physical mechanism behind the strong vortex approximation can be easily understood by analyzing evolution of vorticity. Consideration of an axisymmetric ($\partial/\partial\theta = 0$) bathtub-type flow of inviscid and incompressible fluid is sufficient for our purposes. Analysis of other aspects of evolution of small disturbances in axisymmetric vortical flows can be found in Refs. 4 and 5. In inviscid fluid, the vorticity vectors ω evolve in exactly the same way as the corresponding material line elements[6] and the circulation, defined here as $\gamma \equiv v_\theta r$ satisfies the equation[3,6] $d\gamma/dt = 0$, where d/dt denotes the substantial derivative. Unlike the scalar transport,[7] the vorticity transport does affect the velocity field.[6,8] Let us assume that, initially, the vorticity vector ω_0 and the corresponding material line element (or material vector) AB shown in Fig. 1(a) are directed along z axis ($\omega_r = 0$). Obviously, the value of γ must be the same at A and B. After a short time interval, the position of the same material line element without significant vorticity is shown by $A'B'$. The vorticity component ω_r takes a negative value. Since $r(B') < r(A')$, the rotation at B' is faster. If γ has the same sign as ω_z [negative in Fig. 1(a)] then the vector $A'B'$ has its θ component directed toward the reader. Hence, the flow generates the vorticity ω_θ whose direction is shown in Fig. 1. This vorticity acts to rotate the vector $A'B'$ back to the vertical direction. If $\gamma\omega_0$ is large, a very small deviation from the vertical direction, such as shown by the vector $A'B''$, would be sufficient to generate the vorticity ω_θ required to preserve the initial direction of the vector AB. In this case, the vorticity/velocity interactions adjust the flow in a way that keeps generation of ω_θ under control. This adjustment is not immediate and can be characterized by a certain characteristic time τ_*. If $v_r \sim v_0 = \text{const}$ in the region under consideration, then this process is also characterized by the length scale $\delta_* \sim v_0\tau_*$, which is called here the ''critical scale.'' The condition of $\omega_r \to 0$ requires that v_r does not depend on z as determined by (1). Note that only the case $\gamma\omega_z > 0$, which is most typical, is considered here. A negative value of the product $\gamma\omega_z$ would have the opposite, destabilizing effect on the flow. A more detailed discussion of the stability of this flow can be found in Refs. 9 and 10.

Considering the mechanism of vorticity evolution discussed above, it is quite obvious that approximation (1) is not valid in the vicinity of a disturbance whose characteristic scale is smaller than δ_* since the flow does not have enough time to adjust itself to changing conditions. The validity or falsity of approximation (1) in the far field of this flow, which is the focus of the present work, is not so obvious. We investigate the response of the bathtub-type flow with strong axial vorticity to an axisymmetric disturbance of size δ such as shown in Fig. 1. The undisturbed flow is either uniform or the characteristic length scale of its nonuniformity is much larger than δ and δ_*. The disturbance represents a ring-shaped hump which is placed at the distance r_0 from the axis as shown in Fig. 1. The disturbance is small and located far from the drain: $\delta \ll r_0$. The undisturbed velocity, stream function, vorticity, and circulation in the vicinity of the hump are given by $v_r = -v_0$, $v_z = 0$, $\psi_0 = v_0 r_0 z$, $\omega_z = \omega_0$, $\omega_r = 0$, $\omega_\theta = 0$, and $\gamma = \gamma_0$. The Rossby number is defined as $\mathrm{Rs} = v_0/(\omega_0\gamma_0)^{1/2}$ and the critical scale is introduced by $\delta_* = 2^{1/2} r_0 \mathrm{Rs}$. As required by the strong vortex approximation, the Rossby number is presumed to be $\mathrm{Rs} \ll 1$. When dealing with a real viscous fluid, we assume that its viscosity is sufficiently small so that the thickness of the viscous boundary layer formed at the bottom of the tank is much smaller than the scales δ and δ_*. In this case, the presence of the viscous

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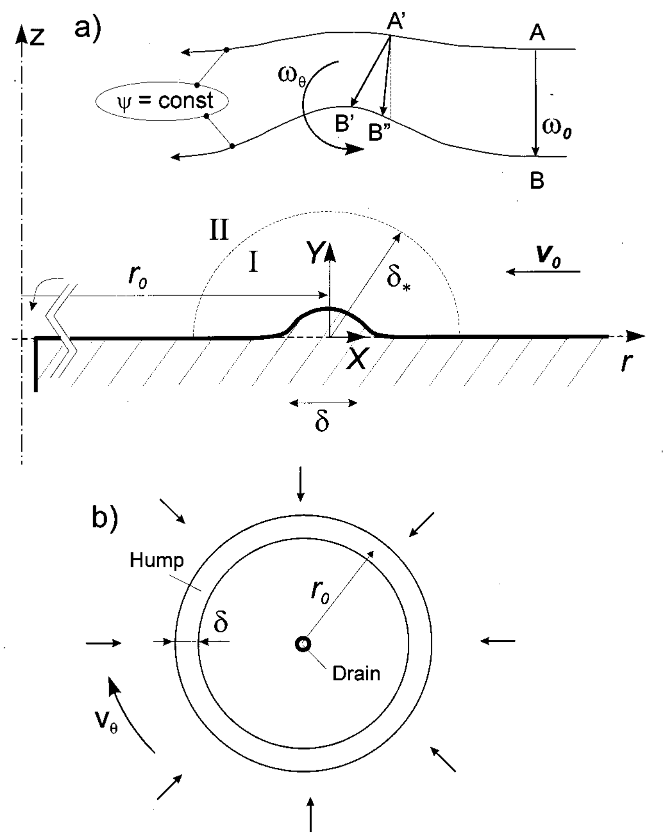

FIG. 1. Schematic of the axisymmeric vortical flow over the hump: (a) the flow image on the $r-z$ plane; (b) view from the top.

boundary layer would effectively result only in a minor change of the shape of the hump.

The general equations determining evolution of vorticity in an axisymmetric inviscid flow are given by[6]

$$\frac{d\omega_z}{dt}=\omega_z\frac{\partial v_z}{\partial z}+\omega_r\frac{\partial v_z}{\partial r},\tag{2}$$

$$\frac{d\omega_r}{dt}=\omega_z\frac{\partial v_r}{\partial z}+\omega_r\frac{\partial v_r}{\partial r},\tag{3}$$

$$\frac{d\omega_\theta/r}{dt}=\omega_z\frac{\partial v_\theta/r}{\partial z}+\omega_r\frac{\partial v_\theta/r}{\partial r}=\gamma\omega\cdot\nabla r^{-2}=-2\,\frac{\gamma\omega_r}{r^3},\tag{4}$$

$$\frac{\partial^2\psi}{\partial z^2}+r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)=-r\omega_\theta.\tag{5}$$

We also note that the following equations: $\omega_r=-r^{-1}\partial\gamma/\partial z$, $\omega_z=r^{-1}\partial\gamma/\partial r$, and $\omega_\theta=\partial v_r/\partial z-\partial v_z/\partial r$ are used in (4) and (5).

The case of a subcritical disturbance $\delta\ll\delta_*$ shown in Fig. 1 is most interesting. The flow is divided into two asymptotic regions: (I) the inner (buffer) zone $r-r_0\sim z\sim\delta$ and (II) the outer (wave) zone $r-r_0\sim z\sim\delta_*$. The characteristic variables of the inner zone are introduced as $X=(r-r_0)/\delta$, $Y=z/\delta$, $\Psi=\psi/(v_0r_0\delta)$, $\Omega_\theta=\omega_\theta(r_0\mathrm{Rs})^2/(v_0\delta)$. The characteristic variable Ω_θ is chosen to ensure that the main terms in (4) are of the same order. The treatment of the inner zone is quite simple. We retain only the leading term in the expansion $\Psi=\Psi_0+\dots$ so that Eq. (5) takes the form $\nabla^2\Psi_0=0$, where ∇^2 denotes the Laplace operator $\partial^2/\partial X^2+\partial^2/\partial Y^2$. Indeed, after neglecting the term $r^{-1}\partial\psi/\partial r$ whose relative contribution is small ($\sim\delta/r_0$), Eq. (5) takes the form $\nabla^2\Psi=-2\Omega_\theta(\delta/\delta_*)^2$. The vorticity generation term appears to be small ($\sim\delta^2/\delta_*^2$) so that the stream function in the inner zone is, to the leading order, potential ($\nabla^2\Psi_0=0$).

While the exact value of Ψ_0 depends on the shape of the hump, its far asymptote (of the inner zone) is given by $\Psi_0=Y-a_0Y/(X^2+Y^2)$ where the constant $a_0\sim1$ is determined by the hump geometry. As $X,Y\to\infty$, the disturbance of the uniform flow becomes small and

$$\psi\to\psi_0+\epsilon\psi_1,\quad \psi_0=v_0r_0\delta_*y,$$
$$\psi_1=-v_0r_0\delta_*a_0\frac{y}{x^2+y^2},\tag{6}$$

where the outer zone variables are introduced as $x=(r-r_0)/\delta_*$, $y=z/\delta_*$, and $\epsilon\equiv(\delta/\delta_*)^2$ is a small parameter.

The outer zone needs a more detailed consideration and, for this zone, we seek the solution in the form of the expansions:

$$\psi=\psi_0+\epsilon\psi_1+\dots,\quad v_r=v_0+\epsilon v_{r1}+\dots,$$
$$v_z=\epsilon v_{r1}+\dots,$$
$$\omega_z=\omega_0+\dots,\quad \omega_r=\epsilon\omega_{r1}+\dots,\quad \omega_\theta=\epsilon\omega_{\theta1}+\dots,$$
$$\gamma=\gamma_0+\dots.$$

The leading order approximations of (3), (4) and (5) take the form

$$v_0\frac{\partial\omega_{r1}}{\partial x}=\omega_0\frac{\partial v_{r1}}{\partial y}=-\frac{\omega_0}{r_0\delta_*}\frac{\partial^2\psi_1}{\partial y^2},\tag{7}$$

$$v_0\frac{\partial\omega_{\theta1}}{\partial x}=-\frac{2\delta_*\gamma_0}{r_0^2}\omega_{r1},\tag{8}$$

$$\frac{\partial^2\psi_1}{\partial y^2}+\frac{\partial^2\psi_1}{\partial x^2}=-r_0\delta_*^2\omega_{\theta1}.\tag{9}$$

By differentiating (9) twice with respect to x and using (7) and (8), we convert this system into a single equation

$$\frac{\partial^4\psi_1}{\partial x^2\partial y^2}+\frac{\partial^4\psi_1}{\partial x^4}=-\frac{\partial^2\psi_1}{\partial y^2}.\tag{10}$$

This equation differs from the Rayleigh beam equation only by the sign of the first term. The substitution of the exponential solution $\exp(ik_xx+ik_yy)$ into (10) yields the dispersion relation for steady waves $k_y^2=k_x^2/(1-k_x^2)$. The exponential solution takes the form $F_k(y,k_x)\exp(ik_xx)$, where

$$F_k(y,k_x)\equiv\begin{cases}\exp(-yf_k(k_x)), & k_x^2>1\\ \cos(yf_k(k_x)), & k_x^2<1\end{cases}$$

and

$$f_k(k_x)\equiv\frac{k_x^2}{|k_x^2-1|^{1/2}}.\tag{11}$$

If $k_x^2<1$, Eq. (10) allows for an arbitrary combination of steady free waves $\sin(k_y y)\exp(ik_x x)$ which satisfy the undisturbed boundary condition $\psi_1=0$ at $y=0$. Here, we are interested in the waves generated by the disturbance and the free waves are excluded from consideration. The response of Eq. (10) to the disturbance can be represented as

$$\psi_1=-\int_{-\infty}^{+\infty} a_1 F_k(y,k_x)\exp(ik_x x)\,dk_x. \quad (12)$$

In general, $a_1=a_1(k_x)$ is not constant and should be determined by matching with (6). For this case, however, the matching conditions are satisfied by constant $a_1=a_0 v_0 r_0 \delta_*/2$. Indeed, the near field of the outer expansion $x,y\to 0$ corresponds to high-frequency asymptote $F_k\to\exp(-y|k_x|)$ as $k_x\to\pm\infty$ whose substitution into (12) yields after evaluation of the Fourier integral $\psi_1\to -2a_1 y/(y^2+x^2)$. In addition, since $F_k(0,k_x)\to 1$ as $y\to 0$ and $x\neq 0$, the stream function ψ_1 determined by (12) satisfies the boundary condition $\psi_1\to 0$ at $y\to 0$ and fixed $x\neq 0$ (actually, for this limit, $\psi_1\to -2\pi a_1\delta_D(x)$, where δ_D denotes Dirac's delta function).

Here, we are concerned with the propagation of the flow disturbances into the main stream. For the far asymptote $(x,y\to\infty)$ of the outer solution, the parts of the integral (12) over the interval $|k_x|>1$, where F_k exponentially tends to zero as $y\to\infty$, can be neglected. Taking into account that f_k is an even function, we obtain

$$\psi_1\to -a_1\int_{-1}^{1}\cos\left(y\left(f_k(k_x)-\frac{x}{y}k_x\right)\right)dk_x. \quad (13)$$

Finally, this integral is calculated for large values of y by the standard asymptotic method which requires evaluation of the integral only in the vicinity of the stationary phase points $k_x=f_\xi(\xi)$, where $f_k'(k_x)=\xi$ and $\xi\equiv x/y$. The solution is represented by the wave

$$\psi_1\to -\frac{1}{2}v_0 r_0 \delta_* a_0\left(\frac{\pi}{y}\right)^{1/2} A(x/y)\cos\left(y\Phi(x/y)+\frac{\pi}{4}\right). \quad (14)$$

The amplitude function $A(\xi)$, the phase function $\Phi(\xi)$, and their asymptotes are shown in Fig. 2 and determined by a cumbersome but fully algorithmic set of equations

$$A(\xi)=\left(\frac{2}{f_k''(f_\xi(\xi))}\right)^{1/2};\quad \Phi(\xi)=f_k(f_\xi(\xi))-\xi f_\xi(\xi),$$

$$f_k(k_x)=\frac{k_x^2}{(1-k_x^2)^{1/2}};\quad f_k'(k_x)=\frac{k_x(2-k_x^2)}{(1-k_x^2)^{3/2}};$$

$$f_k''(k_x)=\frac{(2+k_x^2)}{(1-k_x^2)^{5/2}}, \quad (15)$$

$$f_\xi(\xi)\equiv\mathrm{sign}(\xi)\left(f_\xi^{\bigcirc}(\xi)+\frac{\xi^2+4/3}{\xi^2+1}+\frac{\xi^2+4/3}{3(\xi^4+2\xi^2+1)}\frac{1}{f_\xi^{\bigcirc}(\xi)}\right)^{1/2},$$

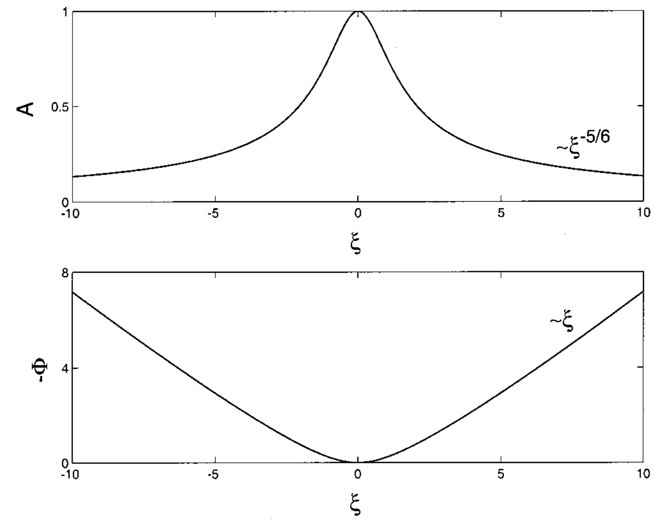

FIG. 2. The amplitude function $A(\xi)$ and the phase function $\Phi(\xi)$ of the wave generated by a subcritical disturbance.

$$f_\xi^{\bigcirc}(\xi)\equiv-\left(\frac{\xi^4+5\xi^2/3+16/27}{2(\xi^6+3\xi^4+3\xi^2+1)}-\frac{\xi(\xi^2+32/27)^{1/2}}{2(\xi^2+1)^2}\right)^{1/3}.$$

Obviously, ψ_1 generated in the far field by a subcritical disturbance does not comply with the strong vortex approximation represented by (1). However, if the disturbance is supercritical $\delta\gg\delta_*$ the stream function is, as it can be expected, consistent with (1). Indeed, in this case, there is only one zone which is determined by the characteristic variables $X=(\delta_*/\delta)x$ and $Y=(\delta_*/\delta)y$. The substitution of X and Y into (10) yields

$$\left(\frac{\delta_*}{\delta}\right)^2\left(\frac{\partial^4\psi_1}{\partial X^2\partial Y^2}+\frac{\partial^4\psi_1}{\partial X^4}\right)=-\frac{\partial^2\psi_1}{\partial Y^2}. \quad (16)$$

The terms on the left-hand side of (16) are, obviously, of smaller order and should be neglected while the resulting equation $\partial^2\psi_1/\partial Y^2=0$ is perfectly consistent with (1).

The main results are now summarized. The critical scale $\delta_*\sim r_0\mathrm{Rs}$, which becomes smaller for faster rotation speeds and smaller Rossby numbers, is introduced. The strong vortex approximation, which is generally valid for small values of the Rossby number, correctly describes the behavior of the flow generated by the disturbance whose characteristic scale is larger than δ_*. However, a more sudden change in the flow induced by the disturbance, whose characteristic scale is smaller than δ_*, is not governed by the strong vortex approximation. At a distance much longer than δ_*, this small disturbance generate a standing wave, which propagates upstream and downstream and does not comply with the strong vortex approximation. The wave is gradually attenuated at larger distances from the disturbance but the attenuation rate is slower than that of a similar disturbance in a potential flow. It was found in Refs. 8, 9, and 10 that the strong vortex approximation may be invalid near the drain of a bathtub-type flow (while the asymptote $\Delta\psi\sim r^{4/3}\Delta z$ is valid near the axis). This is consistent with the present analysis which indicates that a sudden disturbance can make the strong vortex approximation inapplicable.

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