Asymptotic approximations for swirling turbulent plume rising from circular sources

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Governing equations of swirling turbulent buoyant plumes rising from horizontal circular sources into a stationary surrounding are established with the plume function considered. In an attempt to find out the analytical solutions for both lazy and forced plumes, we derive the asymptotic approximations with first-order expansions for all swirling plume variables, including the radius, swirl ratio, axial velocity, and temperature, by applying regular perturbation methods with the swirl term being the perturbative part. Finally, the asymptotic solutions are compared with the numerical evaluations conducted through the fourth-order Runge-Kutta method. The results show that, for lazy plumes, the zeroth-order expansions are good enough to approximate the solutions for each variable, while the first-order expansions are found to match the numerical solution much better for forced plumes, indicating that swirling motions slightly influence lazy plumes but largely affect forced ones. It is also found that, in the presence of swirls, the plume radius slightly increases, while the centerline axial velocity decreases and the temperature barely changes, in both lazy and forced plumes. Additionally, as the input plume function value increases, the swirl ratio decays faster and further decreases the impact on other variables. Especially, a swirl can even turn a moderate forced plume into a lazy plume due to the dominated perturbative part in the near field, which might cause the method for categorizing plumes to be called into question.

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I. INTRODUCTION

Turbulent plumes are widely involved in issues of industrial applications, including chimneys and cooling towers. The present work is extended from our previous studies regarding swirl-enhanced natural draft dry cooling towers [1–4], and it was found that swirling motions enhance the thermal performance of cooling towers by both reducing the adverse cold air inflow and increasing the air flow rate through the towers. Specifically, swirls mitigate the cold air inflow by thinning the boundary layer thickness inside towers and increasing the plume width above towers. The corresponding details inside towers were discussed [2,3], while the mechanism related to how swirls influence hot air plumes above towers yet remains unclarified. In this regard, focusing on the swirl effects on the plume itself, with no crosswind, cold air inflow, and boundary layer effects taken into account, here we consider a swirling turbulent buoyant plume steadily released from an isothermal circular source into a quiescent surrounding, as shown in Fig. 1. The plume rises from a source with an initial radius b_i, arbitrary tangential velocity $u_{\theta,i}$, uniform axial velocity $u_{z,i}$, and temperature T_i, while the environment is unstratified with a uniform temperature T_∞.

Approaches in terms of plume research usually can be categorized into microscopic and macroscopic methods [5]. The former directly seeks numerical solutions of Navier-Stokes equations with

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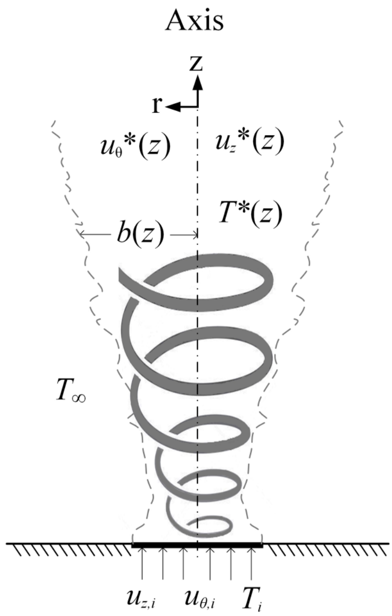

FIG. 1. Schematic diagram of a steady swirling plume rising from a circular source.

different turbulence models through computational fluid dynamics, while the latter first simplifies Navier-Stokes equations based on several hypotheses and then looks for numerical solutions with a much lower cost or makes further modifications attempting to reach the analytical solutions. The macroscopic method is also known as the integral method and is adopted in the current study. The corresponding integral form was first developed by Morton, Taylor, and Turner [6], referred to as the MTT model, which is generally based on assumptions regarding the entrainment rate proportional to the axial velocity, Boussinesq approximation, and the plume properties in the direction orthogonal to the flow direction.

In an attempt to analytically solve the MTT model, similarity solutions can readily be obtained by assuming that plume variables behave as power laws in the z direction. However, they are only valid in the far field of the source, and it is undoubted that similarity solutions are based on the point source assumption, while real sources usually have finite area with an initial radius. Thus, determining the virtual origin to correct the similarity solutions for both lazy and forced plumes has been discussed in many studies [7–9]. In addition to similarity solutions, the Γ approach further simplifies the MTT model, and thus, analytical solutions or approximations, covering both near and far fields of the source, can be yielded under certain conditions [10–12]. The dimensionless parameter Γ was proposed by Morton and Middleton [13] and further developed by Hunt and Kaye [10] as (see Table I for nomenclature)

$$\Gamma(z) = \frac{5Q^2F}{4\alpha M^{\frac{5}{2}}}, \tag{1}$$

where the volume flux Q, buoyancy flux F, and axial momentum flux M are defined as [7]

$$\begin{aligned} Q &= \frac{1}{\pi}\left(2\pi\int_0^\infty u_z r dr\right),\\ F &= \frac{2}{\pi}\left(2\pi\int_0^\infty g\beta\Delta T u_z r dr\right),\\ M &= \frac{2}{\pi}\left(2\pi\int_0^\infty u_z^2 r dr\right). \end{aligned} \tag{2}$$

TABLE I. Nomenclature.

Symbol	Definition
c_p	specific heat capacity [J/(kg K)]
g	acceleration of gravity (m/s^2)
p	pressure (Pa)
r	radius (m)
G	Swirl ratio
T	temperature (K)
u	velocity (m/s)
z	elevation (m)
β	thermal expansion coefficient (1/K)
α	entrainment coefficient
ρ	density (kg/m^3)
ν	kinematic viscosity (m^2/s)
Γ	plume function
∞	ambient
r, θ, z	radial, tangential, axial direction
Ri	Richardson number
b	plume width
e	entrainment
i	initial
Z	dimensionless height
B	dimensionless plume width
U	dimensionless velocity
Θ	dimensionless temperature difference

In accordance with the plume function value, plumes can be categorized into lazy ($\Gamma > 1$), pure ($\Gamma = 1$), and forced ($0 < \Gamma < 1$) plumes, with special flow regimes including jets ($\Gamma = 0$) and fountains ($\Gamma < 0$).

In the presence of swirls, Lee [14] theoretically established governing equations for swirling plumes based on the MTT model with the Gaussian axial velocity and temperature profile assumptions in a cross section of the plume and found the power series solutions in the near field of the source. Narain and Uberoi [15] also gave the similarity solutions, which are valid only in the far field of the source and require the virtual origin correction, as mentioned before. With swirling motions present, however, the virtual origin assumption might be called into question since tangential momenta fall into a singularity there. Given that either the near-field or far-field solution has its restrictions, numerical results were finally illustrated in the pioneering studies subjected to swirling plumes [14,15]. In addition, their investigations did not consider different types of plumes. Note that only the common plumes, lazy and forced ones, will be our concern in this study. These two plumes apparently show different characteristics; that is, plume neck and maximum axial velocity can be found in sufficiently lazy plumes, while they are absolutely absent in forced ones. Hence, analyzing the swirl effects on turbulent plumes depending on the plume types becomes indispensable. Before doing so, it is necessary to emphasize that the determination of the entrainment coefficient α still remains controversial [5], and it is also influenced by swirling motions [15,16] or rotating environments [17]. By retaining the MTT entrainment hypothesis as the benchmark (constant α), the acceptable value for Gaussian profiles, in the absence of rotational momentum, is in the range $0.045 < \alpha_j < 0.056$ in jets and $0.07 < \alpha_p < 0.11$ in plumes [18]. In contrast, with swirl present, this coefficient could increase to 0.179 depending on the swirl intensity [16]. To accurately predict the entrainment, many efforts have been made to correlate an empirical function of the entrainment coefficient in terms of the local Richardson number [19–21]. Those empirical correlations provoke much more complexity in the differential equation system when swirls are considered. Since the aim of this study is to qualitatively and analytically investigate how swirls influence the plume properties for both lazy and forced plumes, the invariant entrainment coefficient model is adopted at this step. A similar clarification was also made in the classic lazy plume theory by Hunt and Kaye [10].

In this study we focus on an analytical investigation of swirl influences on basic properties of both lazy and forced plumes in uniform quiescent surroundings, including plume radius, axial velocity, and temperature. The constant-entrainment-coefficient model is adopted. We first simplify the swirling plume governing equations into a first-order ordinary equation system based on the MTT model and Γ approach in Sec. II. Then the asymptotic solutions are derived in Sec. III for both lazy and forced plumes by applying the regular perturbation method with the swirl term as the perturbative part. A comparison with numerical results is presented, and physical insights are discussed in Sec. IV. Conclusions are drawn in Sec. V.

II. GOVERNING EQUATIONS

The MTT model [6] is adopted, and the approach is also based on the Reynolds decomposition and other further assumptions, as reported in [14,15,22]. The hypotheses in this study, in general, can be summarized as follows:

(1) The flow is steady and axisymmetric.

(2) The entrainment velocity u_e is proportional to the axial velocity at the center of the plume u_z^* (entrainment assumption: $u_e = \alpha u_z^*$).

(3) The radial profiles of the mean axial velocity and buoyancy are similar (Gaussian) at all heights.

(4) The flow is Boussinesq.

(5) The flow is long and narrow for the evocation of the usual boundary layer approximation.

(6) All triple correlations, as well as certain double correlations, are neglected.

(7) Molecular viscosity is neglected in comparison with the eddy one.

By means of these hypotheses, the system of equations describing the turbulent swirling plume can be written as follows: The incompressible continuity equation

$$\frac{\partial (ru_r)}{\partial r} + \frac{\partial (ru_z)}{\partial z} = 0. \tag{3}$$

The radial component of momentum conservation

$$\frac{u_\theta^2}{r} = \frac{1}{\rho_\infty}\frac{\partial p}{\partial r}. \tag{4}$$

The tangential component of momentum conservation

$$\frac{\partial}{\partial r}(r^2 u_r u_\theta) + \frac{\partial}{\partial z}(r^2 u_\theta u_z) = -\frac{\partial}{\partial r}(r^2 \overline{u_r' u_\theta'}). \tag{5}$$

The axial component of momentum conservation

$$\frac{\partial}{\partial r}(ru_r u_z) + \frac{\partial}{\partial z}\left(ru_z^2\right) = -\frac{r}{\rho_\infty}\frac{\partial p}{\partial z} + g\beta(T - T_\infty)r - \frac{\partial}{\partial r}(r\overline{u_r' u_z'}). \tag{6}$$

Energy conservation

$$\frac{\partial}{\partial r}(ru_r T) + \frac{\partial}{\partial z}(ru_z T) = -\frac{\partial}{\partial r}(r\overline{u_r' T'}). \tag{7}$$

The radial profiles of time-mean axial velocity and buoyancy are similar (Gaussian) in the plume at all heights as previously assumed. A similar profile of swirling velocity, however, remains unknown, and it is usually determined experimentally:

$$u_z(r,z) = u_z^*(z)\exp\left(-\frac{r^2}{b^2}\right), \tag{8}$$

$$u_\theta(r,z) = u_\theta^*(z) g\left(\frac{r}{b}\right), \tag{9}$$

$$\Delta T(r,z) = \Delta T^*(z)\exp\left(-\frac{r^2}{b^2}\right), \tag{10}$$

where $b = b(z)$ is the width of the plume at which time-mean axial velocity decreases to e^{-1} of its maximum magnitude u_z^*. It should be emphasized that this plume width, or radius, is not the real one which can be observed in corresponding experiments but is proportional to it, as indicated in [6]. Owing to the symmetry at $r = 0$ and ambient fluid at $r = \infty$, the corresponding boundary conditions are

$$\begin{aligned} u_r^* = u_\theta^* = \frac{\partial u_z}{\partial r} = \overline{u_r' u_\theta'} = \overline{u_r' u_z'} = \overline{u_r' T'} = 0 \quad &\text{at} \quad r = 0,\\ u_\theta^* = u_z^* = p = \overline{u_r' u_\theta'} = \overline{u_r' u_z'} = \overline{u_r' T'} = 0 \quad &\text{at} \quad r = \infty. \end{aligned} \tag{11}$$

We first look into Eq. (4), and the gauge pressure function can be derived by its indefinite integral as

$$p(r,z) = p_\infty - \rho_\infty \int_r^\infty \frac{u_\theta^2}{r} dr, \tag{12}$$

where p_∞ is the pressure of undisturbed ambient fluid. Substituting similar profiles into it and combining it with the axial momentum equation, the gauge pressure term can be eliminated. Then Eqs. (3) to (7) can be integrated from the plume axis $r = 0$ to $r = \infty$ by applying similar profiles [Eqs. (8) to (10)] and boundary conditions [Eqs. (11)] as

$$\begin{gathered} \frac{d}{dz}(u_z^* b^2) = 2\alpha u_z^* b,\\ \frac{d}{dz}\left(u_z^{*2} b^2 - 4\xi u_\theta^{*2} b^2\right) = 2g\beta \Delta T^* b^2,\\ \frac{d}{dz}(u_\theta^* u_z^* b^3) = 0,\\ \frac{d}{dz}(u_z^* g\beta \Delta T^* b^2) = 0, \end{gathered} \tag{13}$$

where ξ is a constant and depends on the swirl velocity profile $g(\frac{r}{b})$ as

$$\xi = \int_0^\infty \frac{r}{b} \int_{\frac{r}{b}}^\infty \frac{g^2\left(\frac{r}{b}\right)}{\frac{r}{b}} d^2\left(\frac{r}{b}\right). \tag{14}$$

A common approach to solve Eqs. (13) is by assuming that the variables behave as power laws in z, which is also known as similarity solutions, but they are valid only in the field far from the source [15]. In addition, power series expansion results near the source have been investigated, but it is obvious that they are valid only in the near field [14]. Both similarity solutions and power series expansion have corresponding restrictions. In this regard, the plume function developed by Hunt and Kaye [7,10] is introduced, and by substituting the Gaussian profiles of axial velocity and buoyancy [Eqs. (8) and (10)] into Eqs. (1) and (2) we arrive at

$$\Gamma(z) = \frac{5g\beta \Delta T^* b}{4\alpha u_z^{*2}}. \tag{15}$$

Note that this is also different from the plume function developed by Michaux and Vauquelin [23] since they assumed the top-hat profiles on the axial velocity and buoyancy, while we adopted Gaussian profiles on the counterparts. A comprehensive comparison between top-hat and Gaussian profile assumptions on plumes can be found in [24]. In addition, Eq. (15) is related to the form of the local Richardson number Ri as

$$\Gamma(z) = \frac{5}{4\alpha}\mathrm{Ri}. \tag{16}$$

The swirl ratio G is defined as

$$G(z) = \frac{u_\theta^*}{u_z^*}; \tag{17}$$

then Eqs. (13) can be reformulated to clearly express the changing rate with z of b, G, u_z^*, and $g\beta\Delta T^*$ as functions of Γ,

$$\begin{aligned}
\frac{db}{dz} &= \frac{4\alpha}{5}\left(10\xi G^2 + \frac{5}{2} - \Gamma\right),\\
\frac{dG}{dz} &= \frac{4\alpha G}{5b}\left(10\xi G^2 - \frac{5}{2} - \Gamma\right),\\
\frac{du_z^*}{dz} &= -\frac{8\alpha u_z^*}{5b}\left(10\xi G^2 + \frac{5}{4} - \Gamma\right),\\
\frac{dg\beta\Delta T^*}{dz} &= -\frac{8\alpha^2 u_z^{*2}}{5b^2}\Gamma.
\end{aligned} \tag{18}$$

Reorganizing Eqs. (18) yields

$$\frac{d\Gamma}{dz} = \frac{4\alpha\Gamma}{b}(10\xi G^2 + 1 - \Gamma). \tag{19}$$

The above equations show the swirling plume flow properties, indicating that the width of the plume and the axial velocity are certainly influenced by swirling motions, and they reduce to the normal form reported in [23] in the absence of swirl ($G = 0$), but the authors have not found the exact analytical solution so far. However, it is noted that ξ is a constant on the magnitude of $O(10^{-1})$ [14,15] and thus can be regarded as a small parameter. In addition, the G value is usually lower than 0.6 since excessively intense swirl would result in the collapse of the Gaussian profile assumption on the axial velocity equation (8), as reported in [25]. Hence, the regular perturbation method is adopted to approximate the asymptotic solutions in the next section.

III. ANALYTICAL APPROXIMATIONS

The initial conditions for all variables are first expressed as follows:

$$b(0) = b_i, \qquad G(0) = G_i, \quad u_z^*(0) = u_{z,i}^*, \quad \Delta T^*(0) = \Delta T_i^*, \quad \Gamma(0) = \Gamma_i. \tag{20}$$

We then make a change in variable $\widetilde{G}$ for a simpler expression as

$$\widetilde{G} = 10G^2, \qquad \widetilde{G}_i = 10G_i^2, \tag{21}$$

and normalize the height, plume radius, axial velocity, and temperature difference as

$$Z = \frac{z}{b_i}, \quad B = \frac{b}{b_i}, \quad U_z^* = \frac{u_z^*}{u_{z,i}^*}, \quad \Theta = \frac{\Delta T^*}{\Delta T_i^*}. \tag{22}$$

Then the swirling plume ordinary differential equation system can be expressed as

$$\begin{aligned}
\frac{dB}{dZ} &= \frac{4\alpha}{5}\left(\xi\widetilde{G}+\frac{5}{2}-\Gamma\right),\\
\frac{d\widetilde{G}}{dZ} &= \frac{8\alpha\widetilde{G}}{5B}\left(\xi\widetilde{G}-\frac{5}{2}-\Gamma\right),\\
\frac{d\Gamma}{dZ} &= \frac{4\alpha\Gamma}{B}(\xi\widetilde{G}+1-\Gamma),\\
\frac{dU_z^*}{dZ} &= -\frac{8\alpha U_z^*}{5B}\left(\xi\widetilde{G}+\frac{5}{4}-\Gamma\right),\\
\frac{d\Theta}{dZ} &= -\frac{2\alpha\Theta}{B}.
\end{aligned} \tag{23}$$

As mentioned above, ξ is a small parameter, so that each variable in Eqs. (23) can be represented in the following expansion:

$$f(z) = f_0(Z)+\xi f_1(Z)+O(\xi^2), \tag{24}$$

with the boundary conditions $f_0(Z=0)=f_i$ and $f_1(Z=0)=0$. As seen, in addition to $\widetilde{G}$, the leading-order term in Eq. (24) refers to plumes without swirling motions involved, while the higher-order term takes responsibility for swirling effects. We substitute Eq. (24) into (23) and collect the terms of zeroth-power ξ (ξ^0) as follows:

$$\begin{aligned}
\frac{dB_0}{dZ} &= \frac{4\alpha}{5}\left(\frac{5}{2}-\Gamma_0\right),\\
\frac{d\widetilde{G}_0}{dZ} &= \frac{8\alpha\widetilde{G}_0}{5B_0}\left(-\frac{5}{2}-\Gamma_0\right),\\
\frac{d\Gamma_0}{dZ} &= \frac{4\alpha\Gamma_0}{B_0}(1-\Gamma_0),\\
\frac{dU_{z0}^*}{dZ} &= -\frac{8\alpha U_{z0}^*}{5B_0}\left(\frac{5}{4}-\Gamma_0\right),\\
\frac{d\Theta_0}{dZ} &= -\frac{2\alpha\Theta_0}{B_0}.
\end{aligned} \tag{25}$$

Note that $B^{-1}=B_0^{-1}-\xi B_0^{-2}B_1+O(\xi^2)$, but this will result in a nonlinear system of the higher-order equations which cannot be solved. Thus, here we approximate $B^{-1}\approx B_0^{-1}$ for first-order equations and collect the terms as ξ^1:

$$\begin{aligned}
\frac{dB_1}{dZ} &= \frac{4\alpha}{5}(\widetilde{G}_0-\Gamma_1),\\
\frac{d\widetilde{G}_1}{dZ} &= \frac{8\alpha}{5B_0}\left(\widetilde{G}_0^2-\widetilde{G}_0\Gamma_1-\widetilde{G}_1\Gamma_0-\frac{5}{2}\widetilde{G}_1\right),\\
\frac{d\Gamma_1}{dZ} &= \frac{4\alpha}{B_0}(\widetilde{G}_0\Gamma_0-2\Gamma_0\Gamma_1+\Gamma_1),\\
\frac{dU_{z1}^*}{dZ} &= -\frac{8\alpha}{5B_0}\left(U_{z0}^*\widetilde{G}_0-U_{z0}^*\Gamma_1+\frac{5}{4}U_{z1}^*-U_{z1}^*\Gamma_0\right),\\
\frac{d\Theta_1}{dZ} &= -\frac{2\alpha\Theta_1}{B_0}.
\end{aligned} \tag{26}$$

We express zeroth-order equations in terms of Γ_0 as

$$\begin{aligned}
\frac{dB_0}{d\Gamma_0} &= \frac{B_0}{5\Gamma_0}\frac{\frac{5}{2}-\Gamma_0}{1-\Gamma_0},\\
\frac{d\widetilde{G}_0}{d\Gamma_0} &= \frac{2\widetilde{G}_0}{5\Gamma_0}\frac{-\frac{5}{2}-\Gamma_0}{1-\Gamma_0},\\
\frac{dU_{z0}^*}{d\Gamma_0} &= -\frac{2U_{z0}^*}{5\Gamma_0}\frac{\frac{5}{4}-\Gamma_0}{1-\Gamma_0},\\
\frac{d\Theta_0}{d\Gamma_0} &= -\frac{\Theta_0}{2\Gamma_0(1-\Gamma_0)},
\end{aligned} \tag{27}$$

so that the solutions in terms of Γ_0 can be collected as follows:

$$\begin{aligned}
B_0 &= \left(\frac{1-\Gamma_i}{1-\Gamma_0}\right)^{\frac{3}{10}}\left(\frac{\Gamma_0}{\Gamma_i}\right)^{\frac{1}{2}},\\
\widetilde{G}_0 &= \widetilde{G}_i\left(\frac{1-\Gamma_0}{1-\Gamma_i}\right)^{\frac{7}{5}}\left(\frac{\Gamma_i}{\Gamma_0}\right),\\
U_{z0}^* &= \left(\frac{1-\Gamma_0}{1-\Gamma_i}\right)^{\frac{1}{10}}\left(\frac{\Gamma_i}{\Gamma_0}\right)^{\frac{1}{2}},\\
\Theta_0 &= \left(\frac{1-\Gamma_0}{1-\Gamma_i}\right)^{\frac{1}{2}}\left(\frac{\Gamma_i}{\Gamma_0}\right)^{\frac{1}{2}}.
\end{aligned} \tag{28}$$

Note that the zeroth-order solutions cover the cases in the absence of swirl ($G=0$). By substituting Eqs. (28) into (26), one can have the change rate of Γ_1 in terms of Γ_0 as

$$\frac{d\Gamma_1}{d\Gamma_0} = \frac{\widetilde{G}_i\Gamma_i}{(1-\Gamma_i)^{\frac{7}{5}}}\frac{(1-\Gamma_0)^{\frac{2}{5}}}{\Gamma_0} + \frac{\Gamma_1}{\Gamma_0}\frac{1-2\Gamma_0}{1-\Gamma_0}, \tag{29}$$

which has the solution

$$\Gamma_1 = \frac{5}{2}\widetilde{G}_i\Gamma_i\Gamma_0\frac{1-\Gamma_0}{1-\Gamma_i}\left[{}_2\mathscr{F}_1\left(\frac{2}{5},2;\frac{7}{5};1-\Gamma_i\right) - \left(\frac{1-\Gamma_0}{1-\Gamma_i}\right)^{\frac{2}{5}}{}_2\mathscr{F}_1\left(\frac{2}{5},2;\frac{7}{5};1-\Gamma_0\right)\right], \tag{30}$$

where ${}_2\mathscr{F}_1$ is the standard hypergeometric function [26].

Now substituting Eqs. (28) and (30) into (26) and noting that

$$\begin{aligned}
\frac{d\left(\frac{\widetilde{G}_1}{\widetilde{G}_0}\right)}{dZ} &= \frac{8\alpha}{5B_0}(\widetilde{G}_0-\Gamma_1),\\
\frac{d\left(\frac{U_{z1}^*}{U_{z0}^*}\right)}{dZ} &= -\frac{8\alpha}{5B_0}(\widetilde{G}_0-\Gamma_1),
\end{aligned} \tag{31}$$

other variables in the first-order equations in terms of Γ_0 can be illustrated as

$$\begin{aligned}
\frac{dB_1}{d\Gamma_0} = {}& \frac{\widetilde{G}_i\Gamma_i^{\frac{1}{2}}}{5(1-\Gamma_i)\Gamma_0^{\frac{3}{2}}}\left(\frac{1-\Gamma_0}{1-\Gamma_i}\right)^{\frac{1}{10}} - \frac{\widetilde{G}_i\Gamma_i^{\frac{1}{2}}{}_2\mathscr{F}_1\left(\frac{2}{5},2;\frac{7}{5};1-\Gamma_i\right)\Gamma_0^{\frac{1}{2}}}{2(1-\Gamma_i)}\left(\frac{1-\Gamma_i}{1-\Gamma_0}\right)^{\frac{3}{10}}\\
&+\frac{\widetilde{G}_i\Gamma_i^{\frac{1}{2}}\Gamma_0^{\frac{1}{2}}}{2(1-\Gamma_i)}\left(\frac{1-\Gamma_0}{1-\Gamma_i}\right)^{\frac{1}{10}}{}_2\mathscr{F}_1\left(\frac{2}{5},2;\frac{7}{5};1-\Gamma_0\right),
\end{aligned}$$

$$
\begin{aligned}
\frac{d\left(\frac{\widetilde{G}_1}{\widetilde{G}_0}\right)}{d\Gamma_0} &= -\frac{2\widetilde{G}_i\Gamma_i}{5(\Gamma_i-1)^{\frac{7}{5}}}\frac{(\Gamma_0-1)^{\frac{2}{5}}}{\Gamma_0^2}\\
&\quad+\frac{\widetilde{G}_i\Gamma_i}{\Gamma_i-1}\left[{}_2\mathscr{F}_1\left(\frac{2}{5},2;\frac{7}{5};1-\Gamma_i\right)-\left(\frac{\Gamma_0-1}{\Gamma_i-1}\right)^{\frac{2}{5}}{}_2\mathscr{F}_1\left(\frac{2}{5},2;\frac{7}{5};1-\Gamma_0\right)\right],\\
\frac{d\left(\frac{U_{z1}^*}{U_{z0}^*}\right)}{d\Gamma_0} &= \frac{2\widetilde{G}_i\Gamma_i}{5(\Gamma_i-1)^{\frac{7}{5}}}\frac{(\Gamma_0-1)^{\frac{2}{5}}}{\Gamma_0^2}\\
&\quad-\frac{\widetilde{G}_i\Gamma_i}{\Gamma_i-1}\left[{}_2\mathscr{F}_1\left(\frac{2}{5},2;\frac{7}{5};1-\Gamma_i\right)-\left(\frac{\Gamma_0-1}{\Gamma_i-1}\right)^{\frac{2}{5}}{}_2\mathscr{F}_1\left(\frac{2}{5},2;\frac{7}{5};1-\Gamma_0\right)\right],\\
\frac{d\Theta_1}{d\Gamma_0} &= -\frac{\Theta_1}{2\Gamma_0(1-\Gamma_0)}.
\end{aligned}
\tag{32}
$$

With the boundary condition $f_1(\Gamma_0=\Gamma_i)=0$, the solutions can be obtained as

$$
\begin{aligned}
\frac{\widetilde{G}_1}{\widetilde{G}_0} &= -\frac{2}{7}\widetilde{G}_i\Gamma_i\left(\frac{1-\Gamma_0}{1-\Gamma_i}\right)^{\frac{7}{5}}{}_2\mathscr{F}_1\left(\frac{7}{5},2;\frac{12}{5};1-\Gamma_0\right)+\frac{\widetilde{G}_i\Gamma_i}{\Gamma_i-1}{}_2\mathscr{F}_1\left(\frac{2}{5},2;\frac{7}{5};1-\Gamma_i\right)\Gamma_0\\
&\quad-\frac{5}{7}\widetilde{G}_i\Gamma_i\left(\frac{1-\Gamma_0}{1-\Gamma_i}\right)^{\frac{7}{5}}{}_2\mathscr{F}_1\left(\frac{2}{5},2;\frac{12}{5};1-\Gamma_0\right)+C_1,\\
\frac{U_{z1}^*}{U_{z0}^*} &= \frac{2}{7}\widetilde{G}_i\Gamma_i\left(\frac{1-\Gamma_0}{1-\Gamma_i}\right)^{\frac{7}{5}}{}_2\mathscr{F}_1\left(\frac{7}{5},2;\frac{12}{5};1-\Gamma_0\right)-\frac{\widetilde{G}_i\Gamma_i}{\Gamma_i-1}{}_2\mathscr{F}_1\left(\frac{2}{5},2;\frac{7}{5};1-\Gamma_i\right)\Gamma_0\\
&\quad+\frac{5}{7}\widetilde{G}_i\Gamma_i\left(\frac{1-\Gamma_0}{1-\Gamma_i}\right)^{\frac{7}{5}}{}_2\mathscr{F}_1\left(\frac{2}{5},2;\frac{12}{5};1-\Gamma_0\right)-C_1,\\
\Theta_1 &= 0,
\end{aligned}
\tag{33}
$$

where

$$
\begin{aligned}
C_1 &= \frac{2}{7}\widetilde{G}_i\Gamma_i{}_2\mathscr{F}_1\left(\frac{7}{5},2;\frac{12}{5};1-\Gamma_i\right)-\frac{\widetilde{G}_i\Gamma_i^2}{\Gamma_i-1}{}_2\mathscr{F}_1\left(\frac{2}{5},2;\frac{7}{5};1-\Gamma_i\right)\\
&\quad+\frac{5}{7}\widetilde{G}_i\Gamma_i{}_2\mathscr{F}_1\left(\frac{2}{5},2;\frac{12}{5};1-\Gamma_i\right).
\end{aligned}
\tag{34}
$$

Finally, substituting the B_0 solution in terms of Γ_0 into Eqs. (25) yields

$$
\frac{d\Gamma_0}{dZ} = \frac{4\alpha\Gamma_i^{\frac{1}{2}}(1-\Gamma_i)}{B_i}\Gamma_0^{\frac{1}{2}}\left(\frac{1-\Gamma_0}{1-\Gamma_i}\right)^{\frac{13}{10}}. \tag{35}
$$

Now the solutions to zeroth- and first-order equations can be obtained once Eq. (35) is solved, except for $B_1(\Gamma_0)$ because difficulties arise when integrating $\Gamma_0^{\frac{1}{2}}(1-\Gamma_0)^{\frac{1}{10}}{}_2\mathscr{F}_1(\frac{2}{5},2;\frac{7}{5};1-\Gamma_0)$. Hence, we shall discuss further approximated solutions for $B_1(\Gamma_0)$ in accordance with initial lazy ($\Gamma_i>1$) and forced ($\Gamma_i<1$) plumes in the next section.

A. Lazy plume

It is known that swirling motions barely influence the plume in the far field since $G\to 0$ as $Z\to\infty$, so the first-order terms are more likely to correct the plume solutions in the near field. For a lazy plume [which may equivalently be regarded as a source excess of buoyancy flux compared to inertia flux according to Eq. (16)], it is clear that $0 \leqslant {}_2\mathscr{F}_1(\frac{2}{5}, 2; \frac{7}{5}; 1-\Gamma_0) < 1$. Then by taking its Taylor series near Γ_i, we obtain

$$ {}_2\mathscr{F}_1\left(\frac{2}{5}, 2; \frac{7}{5}; 1-\Gamma_0\right) \approx {}_2\mathscr{F}_1\left(\frac{2}{5}, 2; \frac{7}{5}; 1-\Gamma_i\right) - \frac{2\left[\Gamma_i^{-2} - {}_2\mathscr{F}_1\left(\frac{2}{5}, 2; \frac{7}{5}; 1-\Gamma_i\right)\right]}{5(1-\Gamma_i)}(\Gamma_0-\Gamma_i); \quad (36) $$

then the first-order equation of B_1 in terms of Γ_0 can be reformed as

$$ \frac{dB_1}{d\Gamma_0} = C_2\frac{(\Gamma_0-1)^{\frac{1}{10}}}{\Gamma_0^{\frac{3}{2}}} + C_3\frac{\Gamma_0^{\frac{1}{2}}}{(\Gamma_0-1)^{\frac{3}{10}}} + C_4\Gamma_0^{\frac{1}{2}}(\Gamma_0-1)^{\frac{1}{10}} + C_5\Gamma_0^{\frac{3}{2}}(\Gamma_0-1)^{\frac{1}{10}}, \quad (37) $$

where

$$ \begin{aligned} C_2 &= -\frac{\widetilde{G}_i\Gamma_i^{\frac{1}{2}}}{5(\Gamma_i-1)^{\frac{11}{10}}}, \\ C_3 &= \frac{\widetilde{G}_i\Gamma_i^{\frac{1}{2}}{}_2\mathscr{F}_1\left(\frac{2}{5}, 2; \frac{7}{5}; 1-\Gamma_i\right)}{2(\Gamma_i-1)^{\frac{7}{10}}}, \\ C_4 &= -\frac{\widetilde{G}_i\Gamma_i^{\frac{1}{2}}}{2(\Gamma_i-1)^{\frac{11}{10}}}\left[{}_2\mathscr{F}_1\left(\frac{2}{5}, 2; \frac{7}{5}; 1-\Gamma_i\right) - \frac{2-2\Gamma_i^2{}_2\mathscr{F}_1\left(\frac{2}{5}, 2; \frac{7}{5}; 1-\Gamma_i\right)}{5\Gamma_i(\Gamma_i-1)}\right], \\ C_5 &= -\frac{\widetilde{G}_i\left[1-\Gamma_i^2{}_2\mathscr{F}_1\left(\frac{2}{5}, 2; \frac{7}{5}; 1-\Gamma_i\right)\right]}{5\Gamma_i^{\frac{3}{2}}(\Gamma_i-1)^{\frac{21}{10}}}, \end{aligned} \quad (38) $$

so that the solution can be yielded as

$$ \begin{aligned} B_1 = {} & C_2\frac{10}{11}(\Gamma_0-1)^{\frac{11}{10}}{}_2\mathscr{F}_1\left(\frac{11}{10}, \frac{3}{2}; \frac{21}{10}; 1-\Gamma_0\right) \\ & + C_3\frac{10}{7}(\Gamma_0-1)^{\frac{7}{10}}{}_2\mathscr{F}_1\left(-\frac{1}{2}, \frac{7}{10}; \frac{17}{10}; 1-\Gamma_0\right) \\ & + C_4\frac{10}{11}(\Gamma_0-1)^{\frac{11}{10}}{}_2\mathscr{F}_1\left(-\frac{1}{2}, \frac{11}{10}; \frac{21}{10}; 1-\Gamma_0\right) \\ & + C_5\frac{10}{11}(\Gamma_0-1)^{\frac{11}{10}}{}_2\mathscr{F}_1\left(-\frac{3}{2}, \frac{11}{10}; \frac{21}{10}; 1-\Gamma_0\right) + C_6, \end{aligned} \quad (39) $$

where

$$ \begin{aligned} C_6 = {} & -C_2\frac{10}{11}(\Gamma_i-1)^{\frac{11}{10}}{}_2\mathscr{F}_1\left(\frac{11}{10}, \frac{3}{2}; \frac{21}{10}; 1-\Gamma_i\right) \\ & - C_3\frac{10}{7}(\Gamma_i-1)^{\frac{7}{10}}{}_2\mathscr{F}_1\left(-\frac{1}{2}, \frac{7}{10}; \frac{17}{10}; 1-\Gamma_i\right) \\ & - C_4\frac{10}{11}(\Gamma_i-1)^{\frac{11}{10}}{}_2\mathscr{F}_1\left(-\frac{1}{2}, \frac{11}{10}; \frac{21}{10}; 1-\Gamma_i\right) \\ & - C_5\frac{10}{11}(\Gamma_i-1)^{\frac{11}{10}}{}_2\mathscr{F}_1\left(-\frac{3}{2}, \frac{11}{10}; \frac{21}{10}; 1-\Gamma_i\right). \end{aligned} \quad (40) $$

Now Eq. (35) can be expressed as

$$ \frac{d\Gamma_0}{dZ} = -\frac{4\alpha\Gamma_i^{\frac{1}{2}}}{(\Gamma_i-1)^{\frac{3}{10}}}\Gamma_0^{\frac{1}{2}}(\Gamma_0-1)^{\frac{13}{10}}, \quad (41) $$

and the solution can be written only in the implicit form:

$$Z(\Gamma_0) = \frac{5(\Gamma_i - 1)^{\frac{3}{10}}}{6\alpha\Gamma_i^{\frac{1}{2}}}\left[\frac{{}_2\mathscr{F}_1\left(-\frac{3}{10}, \frac{1}{2}; \frac{7}{10}; 1-\Gamma_0\right)}{(\Gamma_0 - 1)^{\frac{3}{10}}} - \frac{{}_2\mathscr{F}_1\left(-\frac{3}{10}, \frac{1}{2}; \frac{7}{10}; 1-\Gamma_i\right)}{(\Gamma_i - 1)^{\frac{3}{10}}}\right]. \tag{42}$$

Note that, in the absence of swirl, the heights where a lazy plume reaches its neck and the maximum axial velocity can be explicitly obtained by substituting $\Gamma_0 = \frac{5}{2}$ and $\Gamma_0 = \frac{5}{4}$ into Eq. (42).

B. Forced plume

For a forced plume [which may equivalently be regarded as a source excess of inertia flux compared to buoyancy flux according to Eq. (16)], on the other hand, $0 \leqslant \Gamma_0^{\frac{1}{2}} < 1$. Similarly, by taking its first-order expansion near Γ_i, we obtain

$$\Gamma_0^{\frac{1}{2}} \approx \Gamma_i^{\frac{1}{2}} + \tfrac{1}{2}\Gamma_i^{-\frac{1}{2}}(\Gamma_0 - \Gamma_i) = \Gamma_i^{\frac{1}{2}} + \tfrac{1}{2}\Gamma_i^{-\frac{1}{2}}[(1-\Gamma_i) - (1-\Gamma_0)], \tag{43}$$

so that the first-order equation of B_1 in terms of Γ_0 can be reformed as

$$\begin{aligned}\frac{dB_1}{d\Gamma_0} = {} & C_7\frac{(1-\Gamma_0)^{\frac{1}{10}}}{\Gamma_0^{\frac{3}{2}}} + C_8\frac{\Gamma_0^{\frac{1}{2}}}{(1-\Gamma_0)^{\frac{3}{10}}} + C_9(1-\Gamma_0)^{\frac{1}{10}}{}_2\mathscr{F}_1\left(\frac{2}{5}, 2; \frac{7}{5}; 1-\Gamma_0\right) \\ & + C_{10}(1-\Gamma_0)^{\frac{11}{10}}{}_2\mathscr{F}_1\left(\frac{2}{5}, 2; \frac{7}{5}; 1-\Gamma_0\right),\end{aligned} \tag{44}$$

where

$$\begin{aligned} C_7 &= \frac{\widetilde{G}_i\Gamma_i^{\frac{1}{2}}}{5(1-\Gamma_i)^{\frac{11}{10}}}, \\ C_8 &= -\frac{\widetilde{G}_i\Gamma_i^{\frac{1}{2}}{}_2\mathscr{F}_1\left(\frac{2}{5}, 2; \frac{7}{5}; 1-\Gamma_i\right)}{2(1-\Gamma_i)^{\frac{7}{10}}}, \\ C_9 &= \frac{\widetilde{G}_i\Gamma_i}{2(1-\Gamma_i)^{\frac{11}{10}}} + \frac{b_i\widetilde{G}_i}{4(1-\Gamma_i)^{\frac{1}{10}}}, \\ C_{10} &= -\frac{\widetilde{G}_i}{4(1-\Gamma_i)^{\frac{11}{10}}};\end{aligned} \tag{45}$$

then the solution can be obtained as

$$\begin{aligned} B_1 = {} & -C_7\frac{10}{11}(1-\Gamma_0)^{\frac{11}{10}}{}_2\mathscr{F}_1\left(\frac{11}{10}, \frac{3}{2}; \frac{21}{10}; 1-\Gamma_0\right) - C_8\frac{10}{7}(1-\Gamma_0)^{\frac{7}{10}}{}_2\mathscr{F}_1\left(-\frac{1}{2}, \frac{7}{10}; \frac{17}{10}; 1-\Gamma_0\right) \\ & + C_9\frac{10}{7}(1-\Gamma_0)^{\frac{11}{10}}\left[\frac{4}{11}{}_2\mathscr{F}_1\left(\frac{11}{10}, 2; \frac{21}{10}; 1-\Gamma_0\right) - {}_2\mathscr{F}_1\left(\frac{2}{5}, 2; \frac{7}{5}; 1-\Gamma_0\right)\right] \\ & + C_{10}\frac{10}{17}(1-\Gamma_0)^{\frac{21}{10}}\left[\frac{4}{21}{}_2\mathscr{F}_1\left(2, \frac{21}{10}; \frac{31}{10}; 1-\Gamma_0\right) - {}_2\mathscr{F}_1\left(\frac{2}{5}, 2; \frac{7}{5}; 1-\Gamma_0\right)\right] + C_{11},\end{aligned} \tag{46}$$

where

$$\begin{aligned} C_{11} = {} & C_7\frac{10}{11}(1-\Gamma_i)^{\frac{11}{10}}{}_2\mathscr{F}_1\left(\frac{11}{10}, \frac{3}{2}; \frac{21}{10}; 1-\Gamma_i\right) + C_8\frac{10}{7}(1-\Gamma_i)^{\frac{7}{10}}{}_2\mathscr{F}_1\left(-\frac{1}{2}, \frac{7}{10}; \frac{17}{10}; 1-\Gamma_i\right) \\ & - C_9\frac{10}{7}(1-\Gamma_i)^{\frac{11}{10}}\left[\frac{4}{11}{}_2\mathscr{F}_1\left(\frac{11}{10}, 2; \frac{21}{10}; 1-\Gamma_i\right) - {}_2\mathscr{F}_1\left(\frac{2}{5}, 2; \frac{7}{5}; 1-\Gamma_i\right)\right] \\ & - C_{10}\frac{10}{17}(1-\Gamma_i)^{\frac{21}{10}}\left[\frac{4}{21}{}_2\mathscr{F}_1\left(2, \frac{21}{10}; \frac{31}{10}; 1-\Gamma_i\right) - {}_2\mathscr{F}_1\left(\frac{2}{5}, 2; \frac{7}{5}; 1-\Gamma_i\right)\right].\end{aligned} \tag{47}$$

Now Eq. (35) can be expressed as

$$\frac{d\Gamma_0}{dZ} = \frac{4\alpha\Gamma_i^{\frac{1}{2}}}{(1-\Gamma_i)^{\frac{3}{10}}}\Gamma_0^{\frac{1}{2}}(1-\Gamma_0)^{\frac{13}{10}}, \tag{48}$$

and again, the solution can be written only in the implicit form:

$$Z(\Gamma_0) = \frac{(1-\Gamma_i)^{\frac{3}{10}}}{2\alpha\Gamma_i^{\frac{1}{2}}}\left[\Gamma_0^{\frac{1}{2}}{}_2\mathscr{F}_1\left(\frac{1}{2},\frac{13}{10};\frac{3}{2};\Gamma_0\right) - \Gamma_i^{\frac{1}{2}}{}_2\mathscr{F}_1\left(\frac{1}{2},\frac{13}{10};\frac{3}{2};\Gamma_i\right)\right]. \tag{49}$$

IV. COMPARISON WITH NUMERICAL RESULTS

The numerical solutions to the swirling plume equations, Eqs. (18) and (19), were conducted by adopting the fourth-order Runge-Kutta method. The input constants are listed in Table II.

Figure 2 presents the vertical evolution of swirling lazy plume variables calculated through both numerical and asymptotic methods. The input swirl ratio is fixed at the largest value, $G_i = 0.6$. The solid lines show the numerical results, while the dashed and dotted lines illustrate the zeroth- and first-order expansions. As observed, for lazy plumes, especially the highly lazy plume ($\Gamma_i = 10$), the zeroth-order expansions show good enough approximations, indicating negligible changes caused by swirls in lazy plumes. As mentioned before, $\widetilde{G}$ is on the magnitude of $O(10^{-1})$, while for lazy plumes Γ is not lower than 1, which explains the slight influence of the perturbative part, referring to swirling terms in Eqs. (18), on each of the variables. Especially, the largest deviation between the zeroth- and first-order expansions is found to be the plume radius for the highly lazy plume, as shown in Fig. 2(a). The obvious deviation in the far field is mainly caused by the simplification of Eq. (36) through the first-order Taylor series near Γ_i. In addition, Fig. 2(b) reveals that, as the initial plume function value increases, the swirl ratio decays faster, especially in the near field. This further causes other variables, like the centerline axial velocity U_z^*, to show certain deviations between zeroth- and first-order expansions in the near field. It is also found that the centerline axial velocity U_z^* increases first and then decreases, and the changing rate, as well as the maximum centerline axial velocity, are reduced by swirls. This is because centrifugal forces are present by introducing swirling motions in plumes, leading the edge of the plume to expand outwards, and thus increase the plume radius. Consequently, the axial velocity decreases due to the continuity. Note that there are no first-order expansions on the buoyancy variable Θ due to the absence of the perturbative term in the buoyancy equation in Eqs. (18). Hence, in general, the zeroth-order expansions are good enough to approximate each variable in lazy plumes, unless one is concerned with the swirl influence in the near field precisely.

The vertical evolution of swirling-forced plumes is demonstrated in Fig. 3, with the input swirl ratio fixed at $G_i = 0.6$. As can be seen, excluding the buoyancy variable which is not influenced by the perturbative term, the zeroth- and first-order expansions show larger differences for each variable in forced plumes than in lazy ones, and the first-order expansions show better matches with the numerical results. This is expected since Γ and $10\xi G^2$ are on the same magnitude of $O(10^{-1})$, particularly in the near field, so the perturbative term cannot be neglected anymore. Furthermore, the swirl ratio G decays much slower in comparison with that in lazy plumes, and the lower the input plume function value is, the slower the swirl ratio decays, as demonstrated in Fig. 3(b). Hence, in forced plumes, swirling motions significantly influence the plume properties excluding

TABLE II. Input constants.

Constant	Value	Reference
α (Gaussian)	0.088	[27]
ξ	0.2	[14]

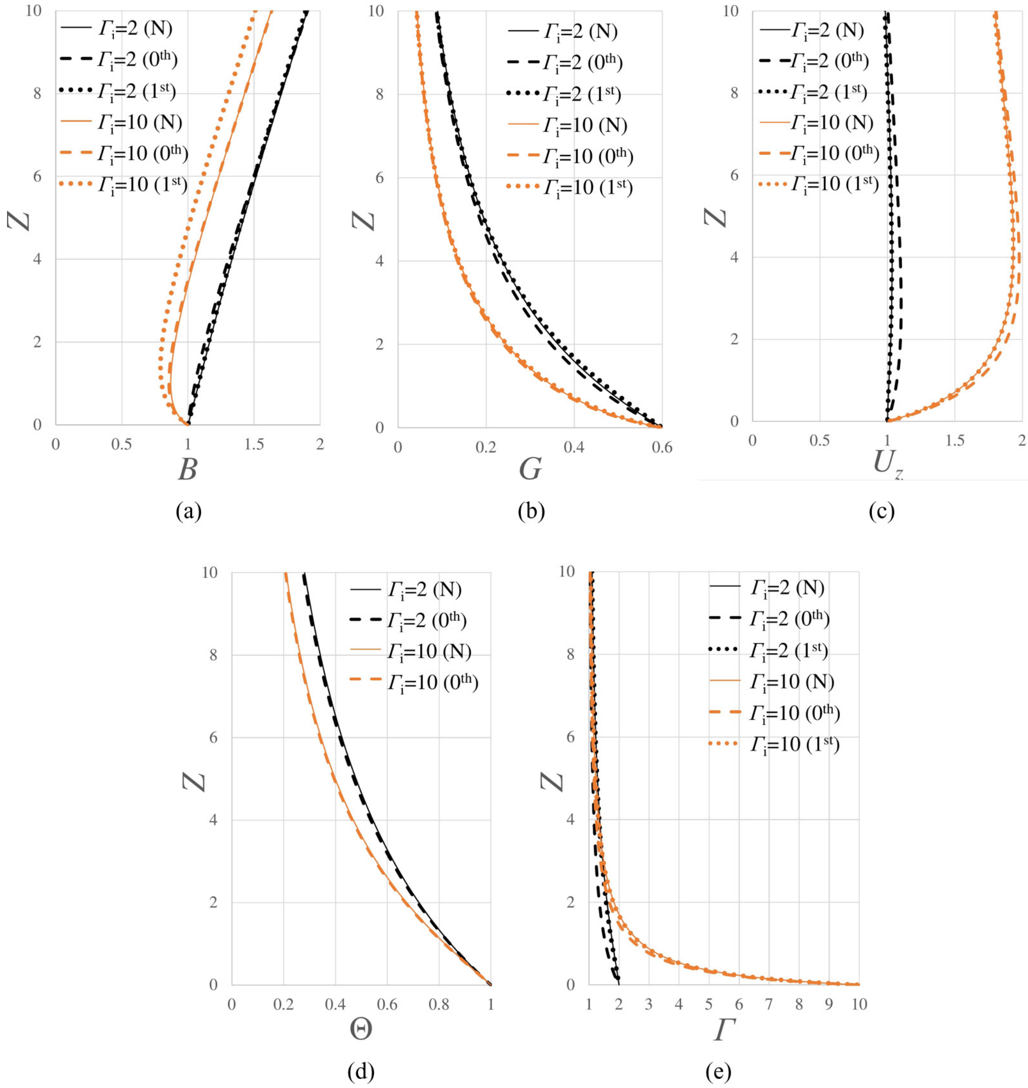

FIG. 2. Vertical evolution of swirling lazy plumes ($G_i = 0.6$). The solid lines are numerical solutions, while the dashed lines and dotted lines represent the zeroth- and first-order expansions of the asymptotic approximations.

buoyancy. The plume radius, as observed in Fig. 3, increases in the near field with swirls present. Additionally, different from the increasing and then decreasing trend on the centerline axial velocity in lazy plumes, it shows a consistently decreasing pattern in forced plumes, and the swirling motion even amplifies the changing rate. The vertical evolution of plume function value for a moderate forced plume ($\Gamma_i = 0.5$) shows an interesting pattern. Specifically, for both numerical and first-order expansion results, the plume function values keep increasing and even exceed 1 and reach their maximum values, then decrease close to 1 in the far field, which means that a moderate forced plume can even be transformed into a lazy plume by introducing swirling motions, as illustrated in Fig. 3(e). This is because, as forced plumes increase close to the pure one ($\Gamma_i \to 1^-$), $(1 - \Gamma_i)$ is much lower than $10\xi G$ in Eqs. (18), resulting in the swirling term dominating the changing rate in the near field. By revisiting the definition of the plume function Γ in our study, as shown in Eq. (15),

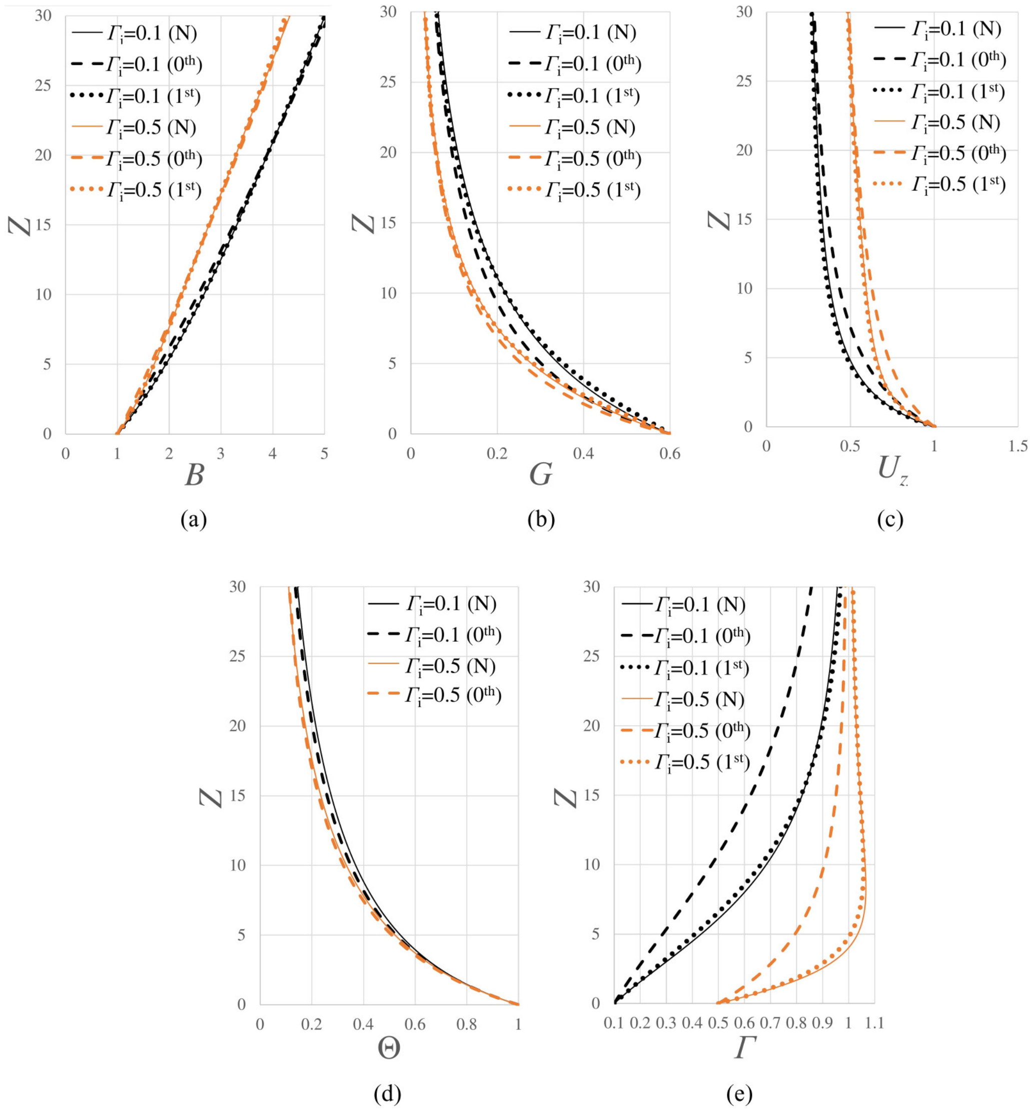

FIG. 3. Vertical evolution of swirling forced plumes ($G_i = 0.6$). The solid lines are numerical solutions, while the dashed lines and dotted lines represent the zeroth- and first-order expansions of the asymptotic approximations.

it is proportional to the plume radius b but inversely proportional to the square of the centerline axial velocity u_z^*. As found before, a swirl always increases the plume radius but decreases the axial velocity in where it has an non-negligible effect (since it decays rapidly), leading to a further increment of the plume function value. Hence, when the initial plume function is less than but not far away from unity, i.e., $\Gamma_i = 0.5$, it tends to increase to 1 as the plume evolves vertically. Once the plume function value is close to 1 and the swirl effects are still present, it exceeds unity and thus turns from a forced plume to a lazy one. Then the plume function value tends to decrease to 1, and we call it the recovery mechanism. At this stage, both the plume function recovery mechanism and swirl effects influence the evolution of the plume function. If the swirl effects outweigh the other mechanism, the plume function will keep increasing and vice versa. When swirl almost decays, the plume function again develops asymptotically to unity in the far field, like how a normal lazy plume behaves. Such a phenomenon should be emphasized since the way to categorize plumes would be called into question in this special regime.

V. CONCLUSIONS

Based on the MTT model and the plume function, the governing equations for swirling turbulent plumes steadily rising from circular sources into a stationary surrounding have been reformed into a nonlinear first-order ordinary differential equation system, including plume radius, swirl ratio, axial velocity, and buoyancy. Due to the difficulty of deriving its exact solutions, we have managed to reach the asymptotic approximations by applying regular perturbation methods, considering the swirling terms as the perturbative part. Therefore, the zeroth-order expansions refer to cases in the absence of swirls, and the exact solutions have been yielded in implicit forms, while the first-order terms are more like corrections, especially in the near field since the swirl ratio tends to zero in the far field. Finally, the approximated analytical solutions were compared with the numerical results found by the fourth-order Runge-Kutta method. The comparisons have shown that, for lazy plumes, the zeroth-order expansions are generally good enough to approximate the solutions for each variable, while the first-order expansions show better matches with the numerical solutions for forced plumes. It has also been found that, in the presence of swirls, the plume radius slightly increases, while the centerline axial velocity decreases and the temperature barely changes, in both lazy and forced plumes. Additionally, as the input plume function value increases, the swirl ratio decays faster and further decreases the impact on other variables. Especially, for a moderate forced plume, it can even be turned into a lazy plume by introducing swirling motions. We therefore believe that establishing a more appropriate plume function with swirl present is the key to the next step before introducing the variable-entrainment-coefficient model which provokes additional complexity.

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APPENDIX: SMALL-PARAMETER ASSUMPTION CLARIFICATION

The regular perturbation method is based on the small parameter ξ defined in Eq. (14), and the value is assumed in Table II. To clarify this, we first assume the tangential velocity profile $g(\frac{r}{b})$ can be assumed to be a third-order polynomial function as

$$g\left(\frac{r}{b}\right) = a_1\frac{r}{b} + a_2\left(\frac{r}{b}\right)^2 + a_3\left(\frac{r}{b}\right)^3, \quad 0 \leqslant \frac{r}{b} \leqslant 1, \tag{A1}$$

with a_1, a_2, and a_3 determined by experimental data in [25]. Furthermore, as indicated in [14], the upper limit of the integration equation (12) can be transformed between the edge of the plume and infinity without introducing much error since the swirl velocity decays rapidly toward the plume edge. Thus, ξ can be estimated by

$$\xi = \int_0^1 \frac{r}{b}\int_{\frac{r}{b}}^1 \frac{g^2(\frac{r}{b})}{\frac{r}{b}} d^2\left(\frac{r}{b}\right) = \frac{1}{6}\left(\frac{1}{2}a_1^2 + \frac{1}{4}a_2^2 + \frac{1}{6}a_3^2 + \frac{2}{3}a_1a_2 + \frac{1}{2}a_1a_3 + \frac{2}{5}a_2a_3\right). \tag{A2}$$

The results with corresponding swirl ratios and coefficients a_1, a_2, and a_3 are shown in Table III.

TABLE III. Values of ξ associated with the swirl ratio and velocity profile coefficients.

G	a_1	a_2	a_3	ξ
0.1168	1.54	2.86	−4.40	0.1620
0.2660	2.425	1.035	−3.46	0.2082
0.4020	6.20	−12.00	5.80	0.2278
0.535	6.04	−10.84	4.80	0.2486

As can be seen, the value of ξ is between 0.16 and 0.25 with a swirl ratio ranging from 0.11 to 0.54. Hence, the value of ξ in this study (0.2) is in the reasonable range and can be regarded as a small parameter for the perturbation method.

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